PB 188 094

STOCHASTIC MODELS FOR BIOCHEMICAL OXYGEN DEMAND AND DISSOLVED OXYGEN IN ESTUARIES

Stephen W. Custer, et al

Virginia Polytechnic Institute
Blacksburg, Virginia

February 1969

CLEARINGHOUSE
FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION

Distributed ... 'to foster, serve and promote the nation's economic development and technological advancement.'

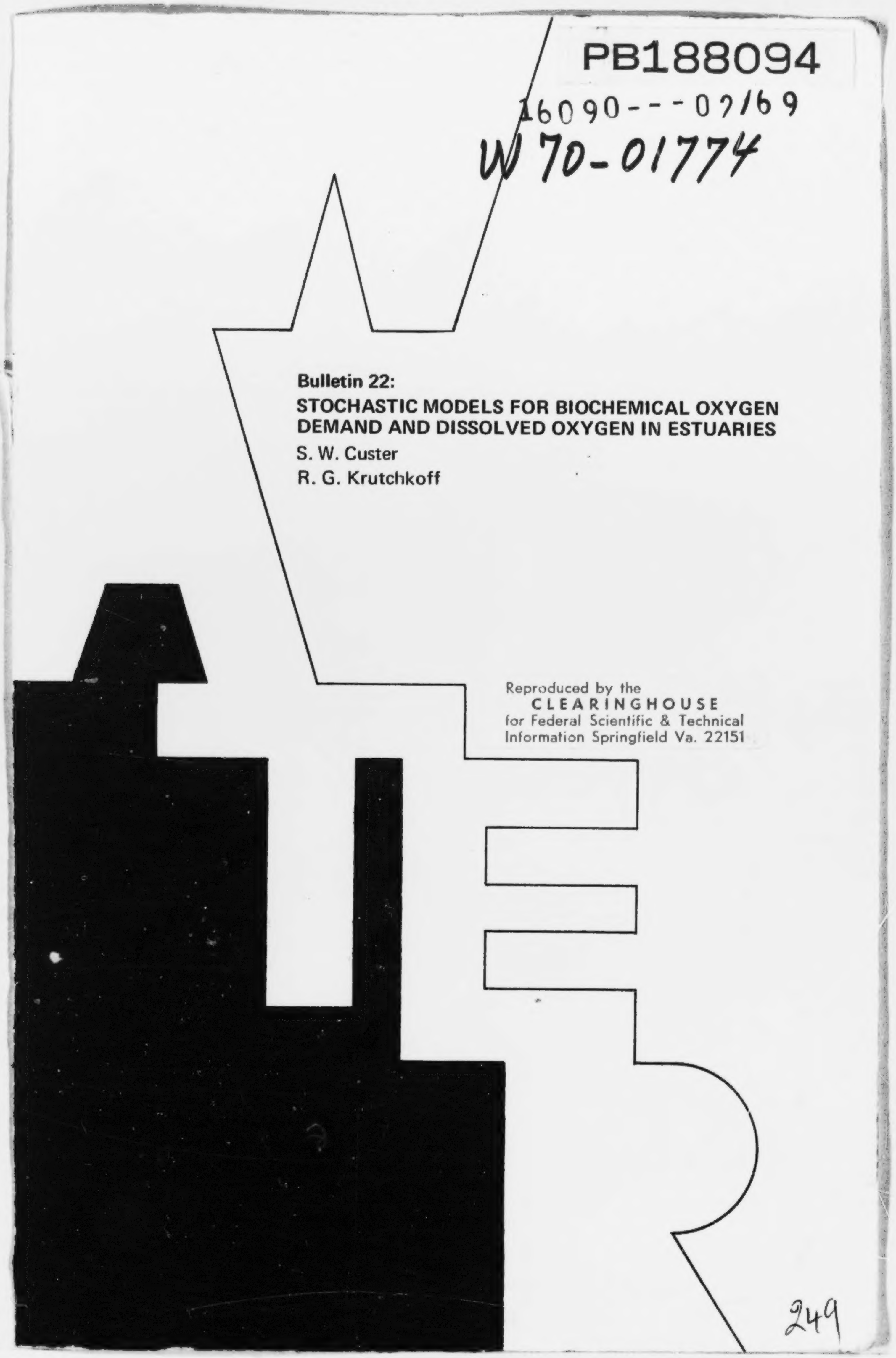
U.S. DEPARTMENT OF COMMERCE/National Bureau of Standards

This document has been approved for public release and sale.

PB188094

16090---07/69

W 70-01774



Bulletin 22:

STOCHASTIC MODELS FOR BIOCHEMICAL OXYGEN DEMAND AND DISSOLVED OXYGEN IN ESTUARIES

S. W. Custer
R. G. Krutchkoff

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va. 22151

249

STOCHASTIC MODELS FOR BIOCHEMICAL OXYGEN DEMAND AND DISSOLVED OXYGEN IN ESTUARIES

Stephen W. Custer
Graduate Student
Department of Statistics

and

Richard G. Krutchkoff
Professor
Department of Statistics

The work upon which this report is based was supported in part by funds provided by the United States Department of the Interior, Office of Water Resources Research, as authorized under the Water Resources Act of 1964.

Water Resources Research Center
Virginia Polytechnic Institute
Blacksburg, Virginia
February 1969

The work upon which this report is based was supported by funds provided by the Federal Water Pollution Control Administration, U. S. Department of the Interior under grant WP-01216 of 1967. The publication of the report was supported in part by funds provided by the Office of Water Resources Research, U. S. Department of the Interior as authorized under the Water Resources Act of 1964.

TABLE OF CONTENTS

INTRODUCTION 1
  General Problem 1
  Scope and Limitations 4

LITERATURE REVIEW 6
  Streeter–Phelps Model 6
  Dobins' Model 7
  Thayer and Krutchkoff's Model 11
  O'Connor's Model 16
  Thomann's Model 17

NON–DISPERSION MODEL 19
  Introduction 19
  Significance of Dispersion 19
  Non–Dispersion Approach 21
  Numerical Results for the Non–Dispersion Model 24
  Probability Distribution for Non–Dispersion Model 26

DIFFUSION MODEL 61
  Introduction 61
  Difference Equations 63
  Continuous Approximations to Difference Equations 69
  Continuous Source 92
  Continuous Approximation to Distribution 95
  Variable Source 100

METHOD OF INTEGRATION 114

LAND RUN OFF AND BENTHAL DEMAND 118
  Difference Equations 118
  Solution to Differential Equations 122
  Application of Section B to BOD and OD Equations 126

EVALUATION OF MODELS 130
  Numerical Results for Diffusion Model 130
  Comparison of Models 131
  Segmenting an Estuary 132

VERIFICATION OF DIFFUSION MODEL 164
  The Potomac Estuary 164
  Evaluation of Parameters 165
  Photosynthesis and Benthal Demand 167
  Application to Potomac Estuary 168
  Conclusions 169

LIST OF TABLES

Table 1. Comparison of Different Methods of Intergration 117

Table 2. Summary of BOD Probability Distribution Five Miles Downstream from the Source 134

Table 3. Summary of OD Probability Distribution Five Miles Downstream from the Source 137

Table 4. Major Waste Water Discharges to the Upper Potomac Estuary 171

Table 5. Water Quality Monitors for the Upper Potomac 172

Table 6. Potomac Estuary Physical Parameters 173

Table 7. Mean K_2 Values Between Stations and Sources 174

Table 8. Mean V_T Values Between Station and Sources 175

LIST OF FIGURES

Figure 1. Theoretical Concentration Distribution in Estuary–Type Flow for Instantaneous, Point Injection 33

Figure 2. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Positive Rush Water 34

Figure 3. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Ebb Tide 35

Figure 4. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Negative Rush Water 36

Figure 5. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Flood Tide 37

Figure 6. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Upstream 38

Figure 7. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Downstream 39

Figure 8. Non–Dispersion Model – Point Source, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Five Miles Downstream 40

Figure 9. Non–Dispersion Model – Point Source,
BOD and OD Concentrations Plotted Against
Time in Tidal Cycle, Distance from Source
Seven Miles Upstream 41

Figure 10. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Distance from Source, Time in Tidal Cycle
-- Positive Rush Water 42

Figure 11. Non–Dispersion Model – Segmented Case,
BOB and OD Concentrations Plotted Against
Distance from Source, Time in Tidal Cycle
-- Ebb Tide 43

Figure 12. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Distance from Source, Time in Tidal Cycle
-- Negative Rush Water 44

Figure 13. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Distance from Source, Time in Tidal Cycle
-- Flood Tide 45

Figure 14. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Time in Tidal Cycle, Distance from Source
Two Miles Upstream .46

Figure 15. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Time in Tidal Cycle, Distance from Source
Two Miles Downstream 47

Figure 16. Non–Dispersion Model – Segmented Case,
BOD and OD Concentrations Plotted Against
Time in Tidal Cycle, Distance from Source
Five Miles Downstream 48

Figure 17. Non–Dispersion Model – Segmented Case, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 49

Figure 18. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Positive Rush Water 50

Figure 19. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Ebb Tide 51

Figure 20. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Negative Rush Water 52

Figure 21. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Flood Tide 53

Figure 22. Non–Dispersion Model – Line Source, Length One Mile , BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Upstream 54

Figure 23. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Downstream 55

Figure 24. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Five Miles Downstream 56

Figure 25. Non–Dispersion Model – Line Source, Length One Mile, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 57

Figure 26. Non–Dispersion Model – Line Source, Length Four Miles, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Flood Time 58

Figure 27. Diffusion Coefficient -- Five Sq. Mi. Per Day O'Connor's Model 59

Figure 28. Non–Dispersion Model -- Line Source, Length Four Miles, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 60

Figure 29. Probabilities for BOD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 0.5 Days 102

Figure 30. Probabilities for OD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 0.5 Days 102

Figure 31. Probabilities for BOD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 1.0 Days 103

Figure 32. Probabilities for OD Particle, Difference Equation Solutions vs. Differential Equation Solutions. Time of Travel is 1.0 Day 103

Figure 33. Probabilities for BOD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 2.0 Days 104

Figure 34. Probabilities for OD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 2.0 Days 104

Figure 35. Probabilities for BOD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 5.0 Days 105

Figure 36. Probabilities for OD Particle, Difference Equation Solutions vs. Differential Equation Solutions, Time of Travel is 5.0 Days 105

Figure 37. Probability Distributions for BOD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 0.5 Days 106

Probability Distributions for BOD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 0.5 Days 107

Figure 39. Probability Distributions for BOD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 1.0 Day 108

Figure 40. Probability Distributions for OD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 1.0 Day 109

Figure 41. Probability Distributions for BOD, Generating Functions Solution vs. Poisson Distribution. Time of Travel is 2.0 Days 110

Figure 42. Probability Distributions for OD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 2.0 Days 111

Figure 43. Probability Distributions for BOD; Generating Functions Solution vs. Poisson Distribution, Time of Travel is 5.0 Days 112

Figure 44. Probability Distributions for OD, Generating Functions Solution vs. Poisson Distribution, Time of Travel is 5.0 Days 113

Figure 45. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle --Positive Rush Water 140

Figure 46. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Ebb Tide 141

Figure 47. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Negative Rush Water 142

Figure 48. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Flood Tide 143

Figure 49. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Upstream 144

Figure 50. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles DDownstream 145

Figure 51. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Five Miles Downstream 146

Figure 52. Diffusion Model -- E is One-Half Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 147

Figure 53. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Positive Rush Water 148

Figure 54. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Ebb Tide 149

Figure 55. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Negative Rush Water 150

Figure 56. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle -- Flood Tide 151

Figure 57. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Upstream 152

Figure 58. Diffusion Model -- E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Downstream 153

Figure 59. Diffusion Model – E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Five Miles Downstream 154

Figure 60. Diffusion Model – E is Two Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 155

Figure 61. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle – Positive Rush Water 156

Figure 62. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle – Ebb Tide 157

Figure 63. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle – Negative Rush Water 158

Figure 64. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Distance from Source, Time in Tidal Cycle – Flood Tide 159

Figure 65. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Upstream. 160

Figure 66. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Two Miles Downstream 161

Figure 67. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Five Miles Downstream 162

Figure 68. Diffusion Model – E is Five Sq. Mi. Per Day, BOD and OD Concentrations Plotted Against Time in Tidal Cycle, Distance from Source Seven Miles Downstream 163

Figure 69. Potomac River Study Area 175

Figure 70. Locations of Sources and Stations, Magnitude of Sources 176

Figure 71. Station One – Observed Hourly Average Dissolved Oxygen Levels 177

Figure 72. Station Two – Observed Hourly Average Dissolved Oxygen Levels 178

Figure 73. Station Three – Observed Hourly Average Dissolved Oxygen Levels 179

Figure 74. Station Four – Observed Hourly Average Dissolved Oxygen Levels 180

Figure 75. Station One – Predicted Mean OD Level and Observed Hourly Average OD Levels 181

Figure 76. Station Two – Predicted Mean OD Level and Observed Hourly Average OD Levels 182

Figure 77. Station Three – Predicted Mean OD Level and Observed Hourly Average OD Levels 183

Figure 78. Station four – Predicted Mean OD Level and Observed Hourly Average OD Levels 184

Figure 79. Station One – Predicted and Observed Cumulative OD Distributions 185

Figure 80. Station Two – Predicted and Observed Cumulative OD Distributions 186

Figure 81. Station Three – Predicted and Observed Cumulative OD Distributions 187

Figure 82. Station Four – Predicted and Observed Cumulative OD Distributions 188

INTRODUCTION

General Problem

The river meets the sea and one of the world's most valuable natural resources is formed. Estuaries represent a natural marriage between the protective but confining terrestrial environment and the uncompromising ocean.

Historically, the community of man clusters on and near the estuaries because they form a natural and extensive transportation system. As witness to this, seven of the ten greatest metropolitan areas in the world have developed near estuaries. The major cities of the United States, for example New York, Los Angeles, Boston, San Francisco, New Orleans, Seattle and Baltimore, owe their existence to their estuaries. Not until comparatively recently, with the advent of rapid land transportation, have sizable pockets of population estranged themselves from the estuary environment. Even today one-third of the population of the United States lives and works near the estuaries (52,9).

In addition to their use for navigation, estuaries serve man in many other ways. They produce bountiful harvest in fish, shellfish, oil and minerals. In our leisure conscious culture they play a major role in offering recreational facilities for swimming, boating and fishing. In many communities estuaries serve as a source of industrial and municipal water supplies. A final use of estuaries is as an ultimate sink for industrial and human waste. Particular attention must be devoted to this use, for improper management here will usurp other beneficial uses.

Estuaries are victims of a conflicting set of circumstances. Pollution producing pockets of population tend to form along the estuaries. Due to their size and turbulence, estuaries give the appearance of being indestructable fields of water which can digest and assimilate vast quantities of waste. In truth estuaries are among the most vulnerable of natural resources. Despite their size the fresh water flow in many estuaries is considerably less than that in most fresh water streams. These estuaries act as natural traps for pollutants. Once a pollutant enters such an estuary it may be carried indefinitely back and forth by the tides, making little progress toward the sea.

Even in estuaries which flush fairly rapidly, pollution can do irreversible damage. The ecology of an estuary is delicately balanced, and a disturbance at any point may cause unpredictable changes in the entire environment. Many species of fish and wildlife use the estuaries to spawn and breed; new life is particularly vulnerable to the intrusion of foreign agents in its environment. Even sub-lethal doses of pollution may retard or prevent natural breeding, eliminating entire generations.

It is not necessary to belabor the detrimental results of pollution on recreational facilities to anyone who has seen the Delaware at Philadelphia or the East River in New York.

The pollutants entering our waterways vary from inorganic compounds which can poison a particular strain of fish, to radioactive substances which threaten an entire population, to organic sewage which can gradually kill the stream or estuary.

Many organic compounds, in particular animal waste and by-products collectively called sewage, are effectively degraded into inoffensive components by the bacteria in natural water. Since the acceptable alternatives are limited it is economically sound to return some of our waste to nature, via streams and estuaries, to let her complete her cycle. As the stream organisms degrade the pollution they require oxygen. The source of the oxygen is the dissolved oxygen in the stream. If the pollution load is heavy, the organisms may deplete the supply of dissolved oxygen and the water becomes anaerobic (43). A stream or estuary with a sufficiently high dissolved oxygen concentration is a gamut of life with insufficient oxygen it becomes stagnant, ordorous, repugnant, and useless.

The analysis in this paper is concerned only with pollutants which adversely effect the dissolved oxygen level of the receiving water. Dissolved oxygen (DO) is measured in terms of its concentration, the usual units are milligrams per liter (mg./l.) or equivalently, parts per million (p.p.m.). Oxygen consuming pollutants are measured in terms of the amount of oxygen required by the bacteria to stabilize them. This is called the Biochemical Oxygen Demand (BOD) and is in the same units as DO.

The water also possesses recuperative power. Oxygen is absorbed at the atmospheric water interface. The oxygen disperses throughout the water by molecular diffusion. Water is a poor solvent for oxygen, causing the DO saturation level to be relatively low. At 20 degrees centigrade saturation occurs at about 9 mg./l. The minimum DO level for maintaining a healthy fish population is listed as about 5 mg./l. DO concentrations below 2 mg./l. will kill almost all types of fish, and if the DO level approaches 0 the water becomes anaerobic (38).

In order to obtain maximum benefit from our water resources it is necessary to carefully manage their use. It would be economically impractical not to utilize the assimilative capacities of our streams and estuaries, at least to the extent that other beneficial uses are not imparied. However, it is insipient management which turns out water resources into open sewers. Since the advantages of using natural waterways as waste receptacles are immediate, while the adverse reprecussions are long range and usually not borne directly by the abuser, it is necessary to use coercive measures to obtain proper water management. The measures take the form of local, state, and federal regulations. Here the legislator is subservient to the engineer. He cannot legislate requirements which are not technically feasible, or standards which are not enforceable.

The prevalent approach is to require each polluter to initiate a specified level of treatment. This has the advantage of being simple to enforce. However, this approach does not consider the capacity of the receiving water. It is probable that even if each contributor treats as required, the total pollution load will over-tax the waterway. On the other hand, the assimilative capacities of the waterway may not be fully utilized, resulting in an unnecessary waste of money. Also, the practice of uniform treatment may be inequitable. Due to differences in waste composition the cost of treatment may vary from pollutor to pollutor. The same net result may be obtained at a smaller total cost by requiring different levels of treatment for different pollutors.

A more sophisticated approach is to predetermine a desired quality for the individual waterways. The entire system of water and pollution sources is then analysed to obtain the most economical means of reaching the desired quality. The advantages of this approach are obvious; however, the analysis may be formidable and expensive.

This systems approach usually sets standards in terms of the average DO level in the waterway. In fact the DO levels vary markedly, and maintaining the desired average levels may not maintain the desired water quality. If the DO concentration falls below the desired level for an extended length of time the water quality may be irreversably damaged. More effective standards can be developed utilizing the entire DO probability distribution, i.e., there is a probability of occurrence associated with each DO concentration level. Clearly, knowledge of the DO probability distribution implies knowledge of the average DO level, but not conversely. The expected percentage of time that the DO would be below any given level can be determined from the DO probability distribution. Water resource managers could use this knowledge to set pollution emission standards which would result in the desired water quality.

Scope and Limitations

For the purpose of mathematical analysis an estuary is defined as that portion of a river which is affected by tidal action (47).

An estuary is a complex physical and ecological environment. In order to proceed with any form of mathematical analysis, it is necessary to make certain idealizations of the physical system. The restrictions we shall make are those which are generally accepted in the literature.

It will be assumed that the pollution and the DO concentrations are uniform in the vertical and lateral direction. Following Kent, such an estuary will be classified as "sectionally homogeneous" (36). For a sectionally homogeneous estuary the analysis is one-dimensional. This restriction is more for practical than theoretical reasons. The analysis could be extended to three dimensions with little theoretical difficulty. However, this would require the field engineer to estimate dispersion characteristics for the vertical and lateral components as well as in the longitudinal direction. This is not practical with present field techniques.

The velocity of the water in the estuary is assumed to be a known function of time. At each location in the stretch of interest the velocity is described by the same function. Even though the velocity may vary in an unpredictable manner due to surface winds, the dominant currents follow a determined pattern. There are no restrictions on the form of this velocity function (other than it be integrable). Tidal velocity can be described fairly accurately by a sinusoidal function with an additional fresh water flow. The fresh water flow may vary from season to season but is considered constant during a given season.

Finally, the cross-sectional area and all the physical parameters are assumed constant throughout the stretch of the estuary under consideration. The requirement of a constant cross-sectional area is not as severe as it may first appear, since in many cases variation in width is compensated by inverse variation in depth. The rate of reaeration is a function of physical parameters which may vary along the course of the estuary (49). Harleman gives evidence that the diffusion coefficient may increase near the mouth of the estuary (21,22). Nevertheless the experience of other researchers is that the above restrictions do not prohibit a useful analysis.

The research in this paper derives the temporal and spatial probability distributions of BOD and DO resulting from a single point source of pollution. This is not an added restriction, since the accumulative effect of several sources can be obtained by the principle of superposition. This principle states that if two or more loads of pollution enter a volume of water the net effect is the sum of the individual effects. The meaning of the term effect will become apparent in the text. In order to apply this principle to oxygen it is necessary to consider the oxygen deficit (OD) rather than the usually considered dissolved oxygen. Oxygen deficity is defined as the difference between the saturation level and the DO level. The principle applies directly for BOD. In probabilistic terms the principle implies that the net distribution is the convolution of the individual distribution.

LITERATURE REVIEW

Streeter-Phelps Model

In 1925 a team of researchers, Streeter and Phelps, presented the first significant mathematical analysis of the assimilative capacities of a stream (56). Their work remains the outstanding single contribution to stream and estuary analysis.

Streeter and Phelps assumed that the dissolved oxygen concentrations in a stream are governed by two independent reactions. First, the oxygen is depleted by the bacteria while stabilizing the organic matter. Secondly, the DO is replenished by absorption at the surface from the atmosphere. By definition the BOD is decreased by the first mechanism at the same rate the DO is depleted. They assumed both of these reactions to the first order. The BOD and DO are decreased at a rate proportional to the remaining BOD, and the DO is increased at a rate proportional to the oxygen deficit. This leads to the following differential equations:

$$\frac{dL}{dt} = -K_1 L \,, \tag{2.1}$$

and

$$\frac{dD}{dt} = K_2 D - K_1 L \,, \tag{2.2}$$

where

t = time of reaction,
L = BOD level,
D = oxygen deficit level,
K1 = coefficient defining the rate of deoxygenation, and
K2 = coefficient defining the rate of reaeration.

The signs are such that K_1 and K_2 are positive.

Initial conditions $L = L_o$ and $D = D_o$ at $t = 0$ are imposed. The differential equations have solutions

$$L = L_o e^{-K_1 t} , \tag{2.3}$$

$$D = \frac{K_1 L_o}{K_2 - K_1} (e^{-K_1 t} - e^{-K_2 t}) + D_o e^{-K_2 t} . \tag{2.4}$$

The independent variable in Equations (2.3) and (2.4) is reaction time, i.e., these equations characterize a static situation such as polluted water in a laboratory beaker. In order to apply the equations to a stream it is necessary to make certain assumptions about the flow. It is assumed that the system is one-dimensional, i.e., complete vertical and lateral mixing. The flow is uniform and steady state and there is no longitudinal dispersion. The last assumption implies that in any cross-section the pollution remains intact, not spilling over into neighboring cross-sections as it moves down stream.

Under these assumptions reaction time is equivalent to travel time. If the uniform velocity is given by

$$U = dx/dt , \tag{2.5}$$

then

$$x = Ut \tag{2.6}$$

where $x = 0$ corresponds to $t = 0$. With this transformation of the independent variable the pollution and DO levels can be obtained from Eqs. (2.3) and (2.4) for any point down stream from $x = 0$.

Dobbins' Model

In a 1964 publication, W. E. Dobbins modified and extended the Streeter-Phelps Equations, taking into account various additional sources of oxygen supply and demand (10). Streeter and Phelps assumed that the only reactions in a stream were the bacterial deoxygenation and reaeration. Dobbins states additional processes which may take place in a particular stretch of the stream.

1. The removal of BOD by sedimentation or absorption.
2. The addition of BOD along the stretch by the scour of bottom deposits or by the diffusion of partially decomposed organic products from the benthal layer into the water above.
3. The addition of BOD along the stretch by the local runoff.
4. The removal of oxygen from the water by diffusion into the benthal layer to satisfy the oxygen demand in the aerobic zone of this layer.
5. The removal of oxygen from the water by purging action of gases rising from the benthal layer.
6. The addition of oxygen by the photosynthetic action of plankton and fixed plants.
7. The removal of oxygen by the respiration of plankton and fixed plants.
8. The continuous redistribution of both BOD and oxygen by the effect of longitudinal dispersion.

Dobbins states equations for BOD and DO concentrations without derivation. A heuristic development of the equations is given by Pyatt (52); O'Connor (47) gives a somewhat more rigid development. The following is a non-rigid outline of the derivation.

Suppose that the concentration of a substance is a function of the spatial parameter x and the temporal parameter t. Let c(x,t) represent the concentration function. Then by definition

$$dc = \frac{\partial c}{\partial x} dx + \frac{\partial c}{\partial t} dt \,, \qquad (2.7)$$

or

$$\frac{dc}{dt} = \frac{\partial c}{\partial x}\frac{dx}{dt} + \frac{\partial c}{\partial t} \qquad (2.8)$$

Letting U = dx/dt represent the longitudinal velocity, one obtains

$$\frac{dc}{dt} = U\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t} \qquad (2.9)$$

dc/dt represents the total change in concentration in time. The total change can be divided into two components; the gain or loss of concentration by diffusion, and the gain or loss of concentration via the local sources and sinks. The change in concentration due to the sources and sinks can be represented by ΣS. The diffusion term is of the Fickian type (11,58), which represents diffusion as proportional to the change in the concentration gradient, i.e.. $E\partial^2 C/\partial x^2$ where E is defined to be the diffusion coefficient. Therefore Equation (2.9) becomes

$$\frac{\partial c}{\partial t} = E\frac{\partial^2 c}{\partial x^2} - U \frac{\partial c}{\partial x} + \Sigma S . \qquad (2.10)$$

Dobbins assumes the process for the stretch as a whole is a steady state process, the conditions in every cross-section being unchanged with time. Therefore, $c(x,t) = c(x)$ and $\partial c/\partial t = 0$. Thus

$$0 = E \frac{d^2 c}{dx^2} - U \frac{dc}{dx} + \Sigma S . \qquad (2.11)$$

The characteristics of the sources and sinks are based on the following assumptions:

1. Bacterial deoxygenation and atmospheric reaeration are treated identically to the Streeter-Phelps model.
2. The removal of BOD by sedimentation, adsorption, and all other non-oxygen consuming processes is a first order reaction; the rate of removal being proportional to the amount present.
3. The removal of oxygen by the benthal demand and by plant respiration, the addition of oxygen by photosynthesis, and the addition of BOD from the benthal layer or the local runoff are all uniform along the stretch.

Therefore the sources and sinks for BOD are mathematically described as follows:

1. L_a, the uniform rate of which pullution is added by local runoff.
2. K_1L, the rate at which pollution decreases due to bacterial action.
3. K_3L, the rate at which pollution decreases due to all non-oxygen consuming processes.

Therefore, the pollution concentration L can be obtained from Equation (2.11), yielding the following differential equation:

$$0 = E\frac{d^2L}{dx^2} - U\frac{dL}{dx} - (K_1 + K_3)L + L_a \, . \tag{2.12}$$

The sources and sinks of oxygen are as follows:

1. K_2D, the rate at which oxygen increases due to reaeration. D is the oxygen deficit.
2. K_1L, the rate at which oxygen decreases due to bacterial action.
3. D_B, the decrease in oxygen due to the benthal demand.

Therefore, the oxygen deficit obtained from Equation (2.11) is

$$0 = E\frac{d^2D}{dx^2} - U\frac{dD}{dx} - K_2D + K_1L + D_B \, . \tag{2.13}$$

In the interest of clarity it is necessary to comment on several of the sources and sinks. The numbers in the following remarks refer to the list of additional processes Dobbins added to the Streeter and Phelps model. The K_3 term includes terms in 2 as well as 1. K_3L is the net effect of all of these processes, so K_3 may be positive or negative depending on the relative magnitudes of the processes. The benthal demand term D_B encompasses all the processes in 4 through 7. In order to include photosynthesis, it is necessary to consider the effect as uniform in time.

The rate of reduction of DO by the benthal demand is assumed to be independent of the oxygen level. Clearly this cannot be the case when DO is near zero, or negative oxygen concentration would result. In truth this assumption loses its validity for DO levels below 1 mg./l. As a precaution our analysis will be restricted to situations where a minimum of 1 mg./l. of oxygen exists.

The longitudinal diffusion coefficient is in many cases uncertain. Dobbins calculated the amount of error which would be introduced if it were neglected. Using extreme values for the stream parameters he shows that the ratio of the true concentration to the concentration obtained by neglecting diffusion is 0.996. Lynch (44) later corrected this value to 0.960, i.e., an error of 4%. In any case, this represents the probable maximum error. Thus, the effect of longitudinal diffusion in a stream is negligible.

The solutions to Equations (2.12) and (2.13) with E = 0 are

$$L = L_o e^{-(K_1+K_3)t} + \frac{L_a}{K_1+K_3}(1-e^{-K_1+K_3)t}) , \tag{2.14}$$

and

$$D = \frac{K_1}{K_2-(K_1+K_3)} (L_o - \frac{L_a}{K_1+K_3})(e^{-(K_1+K_3)t} - e^{-K_2t})$$

$$+ D_o e^{-K_2t} + (\frac{D_B}{K_2} + \frac{K_1 L_a}{K_2 (K_1 + K_3)})(1-e^{-K_2t}) , \tag*{15)}$$

where t is the travel time given by t = x/U, and the initial conditions are $L = L_o$ and $D = D_o$ at t = 0.

If all the parameters except K_1 and K_2 are set equal to zero, Equations (2.14) and (2.15) reduce to the Streeter-Phelps Equations (2.3) and (2.4).

Thayer and Krutchkoff's Model

The Streeter-Phelps and Dobbins models are deterministic; given a set of parameters, BOD and DO values are determined for any point in space. In practice BOD and DO measurements exhibit considerable variation. Relatively accurate predictions of the mean levels of BOD and DO are obtained by the earlier models, but the possible variations are not considered. In order to obtain a realistic appraisal of the resulting water quality it is frequently necessary to anticipate the portion of time that the DO level will fall below certain critical concentrations. This cannot be accomplished using the deterministic model.

In 1966 Thayer and Krutchkoff developed a stochastic model for BOD and DO in streams, (59,60). Their work was based on Dobbins' model, with the mechanisms effecting BOD and DO concentrations modified so as to be random in nature.

Thayer and Krutchkoff made one additional assumption. The pollution and dissolved oxygen were discretized. The term Δ was introduced to represent the optimum unit size. All transitions in BOD and DO were asserted to be in units of size Δ Δ is a measurable parameter of the individual stream. Methods of estimating Δ will be given later.

A change of size Δ in the concentration constitutes a change of one "state." The pollution state m corresponds to a concentration of L_m and the relationship between the two is

$$\text{State } m = L_m/\Delta \ . \tag{2.16}$$

Similarly, OD state n corresponds to concentration $n\Delta = D_n$.

Dobbins' assumptions concerning the precesses in the stream were replaced by their stochastic analogues. In a small time increment h, the pollution or OD can be increased or decreased by Δ or a multiple of Δ by each of the mechanisms considered by Dobbins. The mechanisms are assumed to act independently. The order of magnitude of h will be such that h^2 will be negligible in comparison to h. The probability of change in state is assumed proportional to h, thus a change of more than one state by a single factor in time h has probability o(h). The probability of one change in state for each of the factors is as follows:

1. The expected amount of increase in pollution due to L_a in the time interval h is $L_a h$. Letting p = Pr [Pollution increases by amount Δ due to L_a in time h], one obtains the expected increase due to L_a in time h to be $\Delta p + o(h) + 0 \cdot (1-p-o(h))$.

Therefore, $p = \dfrac{L_a h}{\Delta} + o\,(h)$. By similar arguments:

2. Pr [Pollution decreased by an amount Δ due to K_3 in time h] =

$$\frac{K_3 L_m h}{\Delta} + o(h) .$$

3. Pr[Pollution decreases and OD increases by an amount Δ due to K_1 in time h]

$$= \frac{K_1 L_m h}{\Delta} + o(h) .$$

4. Pr[OD increases by an amount Δ due to D_B in time h] $= \frac{D_B h}{\Delta} + o(h)$.

5. Pr[OD decreases by an amount Δ due to K_2 in time h] $= \frac{K_2 D_n h}{\Delta} + o(h)$.

The joint probability of finding the system with the pollution in state m and the oxygen deficit in state n at time t is denoted by $p_{m,n}(t)$. Combining the factors 1 through 5 one obtains the difference equation

$$P_{m,n}(t+h) = p_{m,n}(t) \left[1 - \frac{L_a h}{\Delta} - \frac{K_3 L_m h}{\Delta} - \frac{K_1 L_m h}{\Delta} - \frac{D_B h}{\Delta} - \frac{K_2 D_n h}{\Delta}\right] +$$

$$p_{m-1,n}(t) \frac{L_a h}{\Delta} + p_{m+1,n}(t) K_3(m+1) h + p_{m+1,n-1}(t) K_1 (m+1)h +$$

$$p_{m,n-1}(t) \frac{D_B h}{\Delta} + p_{m,n+1}(t) K_2 (n+1)h + o(h) . \qquad (2.17)$$

Let P(s,r,t) be the probability generating function of $p_{m,n}(t)$, i.e.,

$$P\,(s,r,t) = \sum_{m=0} \; \sum_{n=0} \; P_{m,n}(t)\, s^m\, r^n \,. \qquad (2.18)$$

Taking limits as h approaches 0 the following differential equation in P(s,r,t) is derived:

$$\frac{\partial \ln P}{\partial t} + K_2\,(r-1)\,\frac{\partial \ln P}{\partial r} - K_3\,(1-s) + K_1\,(r-s)\,\frac{\partial \ln P}{\partial s}$$

$$= L_a/\Delta\,(1-s) + D_B/\Delta\,(1-r) \;. \qquad (2.19)$$

Thayer and Krutchkoff obtained analytic solutions for P (s,r,t) for several sets of initial conditions. The solutions corresponding to the initial condition that the BOD concentration is L_o and the OD is D_o at t = 0 will be useful in our applications. For these initial conditions the solution to Equation (2.19) is

$$P\,(s,r,t) = \exp\Big[\, \frac{-L_aK_1}{\Delta\,(K_1+K_3)\,(K_2-(K_1+K_3)\,)}\,(1-e^{-(K_1+K_3)t}) \;.$$

$$(1-r) + \frac{L_aK_1}{\Delta\, K_2(K_2-(K_1+K_3)\,)}\,(1-e^{-K_2t})\,(1-r) \;-$$

$$\frac{L_a}{\Delta\,(K_1+K_3)}\,(1-e^{-(K_1+K_3)t})\,(1-s) + \frac{D_B}{\Delta K_2}\,(1-e^{-K_2t}\,)\,(1-r)\;\Big] \;.$$

$$[1-(1-r)\,e^{-K_2 t}]\,D_o/\Delta\,.\,[1-\frac{K_1}{K_2-(K_1+K_3)}\,(e^{-(K_1+K_3)t}-e^{-K_2 t})\;.$$

$$(1-r)\;-e^{-(K_1+K_3)\,t}\,(1-s)\;]\;L_o/\Delta\;. \tag{2.20}$$

Equation (2.20) can be put in somewhat more manageable form if several reoccurring groups of symbols are redefined.

$$\alpha_1\,(t) = e^{-(K_1+K_3)\,t}$$

$$\beta_1\,(t) = 1-\alpha_1$$

$$\alpha_2\,(t) = \frac{K_1}{K_2-(K_1+K_3)}\;(e^{-(K_1+K_3)\,t}-e^{-K_2\,t})$$

$$\beta_2\,(t) = 1-\alpha_2$$

$$\alpha_3\,(t) = e^{-K_2\,t}$$

$$\beta_3\,(t) = 1-\alpha_3$$

$$\gamma = \frac{L_aK_1}{\Delta(K_2-(K_1+K_3))}$$

Rewriting Equation (2.20) gives

$$P(s,r,t) = \exp \left\{ (1-r)\left[\frac{\gamma\beta_1}{K_1+K_3} + \frac{\gamma\beta_1}{K_2} + \frac{D_B\beta_3}{\Delta K_2} \right] - \frac{L_a\beta_1}{\Delta(K_1+K_3)}(1-s) \right\} \circ \left[1-(1-r)\,a_3\right]^{D_o/\Delta} \cdot \left[1-(1-r)a_2-(1-s)a_1\right]^{L_o/\Delta}. \tag{2.21}$$

Thayer and Krutchkoff obtained the marginal generating functions for BOD and OD from Equation (2.21) by setting $r = 1$ and $s = 1$, respectively. They were able to pick out the coefficient of s^m and r^n. The coefficients are functions of t. These coefficients give the complete pollution and oxygen deficit distributions. The means of the distributions were identical to Dobbins' equations; and the variances were linear functions of Δ.

O'Connor's Model

The most prominent work in estuary analysis is being done by Donald O'Connor at Manhattan College (46,47,48). His basic approach has been to apply Equations (2.13) and (2.14) to the estuary situation with $E \neq 0$. U is taken as the constant fresh water velocity. Thus the estuary is described as a stream with non-negligible dispersion. The effect of time dependent velocity is not taken into account.

In the derivation of Equations (2.12) and (2.13) E is a measure of the longitudinal dispersion. O'Connor uses E as a "garbage can" component, i.e., all effects not taken into account by other terms in the equations are

attributed to E. The most prominent of these terms is the tidal velocity. O'Connor estimates E by using Equations (2.12), (2.13) and estuary data; therefore, it is reasonable to expect that if all stray components are placed in the estimate of E, these effects will be collectively felt when E is used for predictive purposes. If the effect of longitudinal dispersion is measured independently and compared to E as estimated by O'Connor the difference may be 10-fold or larger.

It may be noted that Equations (2.12) and (2.13) are functions of space only. However, it is clear that the concentrations in an estuary are also functions of time in the tidal cycle. For this reason O'Connor recommends data collection by the method of "slack water," i.e., data for a given location is only collected when the tidal velocity is zero. This method, or any method which only considers a fixed time in the tidal cycle, will mask the time dependent nature of the concentrations.

Thomann's Model

Robert Thomann has presented a systems analysis approach which describes the time variable nature of the dissolved oxygen in an estuary (61,62,64). He assumes that the estuary can be segmented into a discrete number of sections. The DO concentrations may vary from segment to segment but are homogeneous within each segment. The rate of change of DO concentration in the kth segment will depend on (a) the net volume transported from segment k-1 into k and from k into k+1; (b) dispersion from both k-1 and k+1 into k; and (c) sources and sinks in segment k.

Thomann assumes that the rate of change of oxygen concentration is dependent upon the concentration differences between segment k and adjacent segments. The dispersion is assumed to be directly proportional to the differences in the concentration. The differential equation for the kth segment is given by

$$V_k \frac{dc_k}{dt} = Q_{k-1,k} \frac{(c_{k-1}+c_k)}{2} - Q_{k,k+1} \frac{(c_k+c_{k+1})}{2} +$$

$$E_{k-1,k}\,(c_{k-1}-c_k) + E_{k,k+1}\,(c_{k+1}-c_k) + V_k f_k(t). \qquad (2.22)$$

where

c_k	is the DO concentration in segment k;
V_k	is the volume of segment k;
$Q_{k-1,k}$	denotes the net flow from segment k-1 to k;
$E_{k-1,k}$	is the dispersion coefficient between segments k-1 and k;
$f_k(t)$	denotes all forcing functions acting in segment k that cause either an increase or decrease in oxygen, i.e., sources and sinks.

A set of equations like (2.20) is obtained, one for each segment of the estuary.

It may be noted that Equation (2.22) is the finite difference form of Equation (2.10). The advection term $Q_{k-1,k}$ is not a function of time. Thus, the time variable velocity is not considered; this is again congregated with $E_{k-1,k}$.

The advantage of this approach is that the advection and dispersion are discrete functions of distance, and the sources and sinks are functions of space and time. The disadvantage is in the number of parameters to be estimated. In applications, Thomann assumes Q and E to be constant.

NON-DISPERSION MODEL

Introduction

The temporal and spatial variations in BOD and DO concentrations observed in an estuary arise from two totally different sources. The first cause of the variation is the random nature of the mechanisms which act upon the pollutants. These random mechanisms are similar to those encountered in stream analysis. The second source of variation is produced by the non-random velocity changes in an estuary. This variation is over and above the variation which can be expected in a stream.

The random component has been considered by Thayer and Krutchkoff, and a similar analysis is applicable to an estuary. The second component is not random and can be handled deterministically. The velocity of an estuary is a function of time; therefore, unlike a stream, the mean BOD and DO levels are functions of time as well as space.

The purpose of this analysis is to predict the variation of the BOD and DO concentrations over small intervals. Therefore, a method such as O'Connor's, which considers only one point in the tidal cycle, will be inadequate. In this section the opposite approach is investigated, i.e., longitudinal dispersion is neglected, and only the effect of the time dependent tidal velocity is considered. Thus, the concentrations will be functions of time and space. The stream equations of Dobbins can be used to obtain the mean BOD and DO concentrations at any point in space and time. The stochastic results of Thayer and Krutchkoff could then be applied to obtain the complete probability distribution.

Significance of Dispersion

The degree of success of this model will, of course, depend upon the size of E. If E is large, this approach is doomed to failure; if on the other hand, the true dispersion coefficient (without the effect of tidal velocity) is of the same order as dispersion in a stream, a comparatively high degree of accuracy could be expected. Dobbins gives numerical examples which illustrate that a dispersion coefficient as high as 1.57 mi.^{2}/day may be neglected in a stream without serious loss of accuracy (10).

The normal range of values of E, as estimated by O'Connor, is 8-10 mi.^{2}/day. In a personal communication with the authors, in March of 1968, Dr. O'Connor estimated that the value of E would be decreased to 2-5 mi.^{2}/day if the effect of tidal velocity were considered separately. Admittedly, if E is as high as 5 mi.^{2}/day serious error would be introduced if it were neglected.

The methods of estimating dispersion are by no means uniform and lead to inconclusive results. Hetling and O'Connell describe several analytic methods of estimating E, and extrapolate the results to estimate the dispersion in the Potomac River (29,30). The dominant factor in each of these methods is the tidal dispersion. There is no published work concerning the relative magnitude of E when the tidal velocity is treated separately; however, it will be considerably smaller than values obtained by conventional methods.

The first approach considered by Hetling and O'Connell is the "Four-Thirds Law," wherein the dispersion coefficient is proportional to the four-thirds power of the "scale of the diffusion phenomena." In estuary applications the characteristic scale is taken to be the length of a tidal excursion. Using this formula, the dispersion coefficient was calculated for a short stretch of the Potomac in Washington, D. C. The values of E obtained ranged from about 5 to 6.5 mi.^{2}/day.

A second formula considered was formulated by Diachishin (8). He approaches the diffusion phenomenon as a random walk process, and develops equations similar to the classical diffusion equations. He concluded that E should be proportional to the square of the length of each random step. The step length was assumed to be equivalent to the length of the tidal excursion. Using this approach, values of E ranging from 12 to 25 mi.^{2}/day are obtained for the same stretch of the Potomac.

Harlemann (21,22), following Taylor's work (57) on diffusion for turbulent flow in pipes, forms analogous expressions for estuary dispersion. These formulae also rely upon the tidal effect. Applying these equations to the Potomac, Hetling and O'Connell obtain values of E in the range of 0.05 to 0.08 mi.^{2}/day.

Bowden, following a similar approach with slightly different assumptions, derived equations which yield dispersion coefficients of 0.006 to 0.01 mi.^{2}/day for the same stretch of the Potomac (29).

In an attempt to clarify the conflicting results, Hetling and O'Connell evaluated the dispersion empirically from existing salinity data. The empirical estimate of E was in the range of 3 to 6 mi.^2/day. However, in a dye dispersion test conducted at a later time the value of E for this stretch of the Potomac was 0.2 mi.^2/day (26,27).

Harlemann stated that his low estimates were valid for the fresh water portion of the estuary, but that density effects in the brackish portions inflate the dispersion. Hetling and O'Connell conclude that the "Four-Thirds Law" yields dispersion coefficients of the proper order of magnitude in brackish water, but caution should be exercised in the fresh water portion of the estuary. Confronted with this array of seemingly incompatible evidence, it appears that estuary engineers are far from having a firm grasp on the complex dispersion phenomenon.

The dispersion coefficient obtained by each of the above approaches is equivalent to the coefficient that would be obtained by the slack water technique. Figure 1 illustrates how a small dispersion effect can lead to a large value of E when the method of slack water is used (23). The figure illustrates the temporal distribution downstream from an instantaneous dye release. At some time after the release the mass of dye passes the observer for the first time. As the tide reverses, the same mass again passes the observer moving upstream. These observations are represented by the first two peaks. At the next low water the dye mass again passes the observer and returns; the distribution of the dye has spread somewhat due to dispersion. This process continues until the dye moves out of range of the observer. If data were taken only at slack water the distribution observed would be the broken line. Obviously, this curve would lead to a very large dispersion coefficient, while in fact, the dispersion is represented by any one of the smaller solid curves.

In light of the evidence cited earlier, the effect of neglecting dispersion in an estuary can not be determined a priori. The only alternative is to develop the non-dispersion model and compare the predicted results with field data. In this way the loss suffered by neglecting dispersion can be compared to the gain realized by including the time variable velocity.

Non–Dispersion Approach

Consider a continuous source of pollution with pollution entering at a rate W (W may be a function of time.). The estuary has velocity U(t). Let L(x,t) be the concentration of BOD at a distance x downstream from the source (x may be positive or negative, negative x represents distance upstream at time t. By specifying x and t a particular cross sectional area of water is defined. Since dispersion is neglected, this area maintains its identity in time, and may be treated as if isolated from the surrounding water. The past history of this area of water is completely determined by U(t). The location x' of the water at any past time t_i is given by

$$x' = x - \int_{t_i}^{t} U(\tau)\, d\tau \quad . \tag{3.1}$$

The source is located at $x' = 0$, thus one can obtain a record of all times in the past that the given area passed the source by obtaining the roots of Equation (3.1) which are less than t. If the velocity has a net positive component there will be a finite number of roots. For convenience let the solution to Equation (3.1) be represented by a vector $T = [t_1, t_2, \ldots t_k]$, where t_i is the time of the i^{th} passing.

Each time the area of water passes the source a concentration of pollutants will be added. The effect of each of these shots of pollution can be calculated by Dobbin's equations where the time parameter is $(t-t_i)$.

As a segment of water of length Δx passes the source the concentration of pollution added to the segment will be

$$c_i = \frac{W \Delta t}{A \Delta x} \quad , \tag{3.2}$$

where

A is the cross sectional area of the estuary, and

Δt is the time required for the segment to pass the source.

Therefore, as $\Delta x \rightarrow 0$,

$$c_i = \frac{W}{A\, U(t_i)} \quad . \tag{3.3}$$

Using c_i as the initial BOD level, and zero as the initial OD level, the resulting BOD and OD concentrations can be calculated for each loading by use of Dobbins' equations. These are represented by $L = [L_1, L_2, \ldots L_k]$ and $D = [D_1, D_2, \ldots D_k]$ respectively. Applying the principle of superposition, the net results are

$$L(x,t) = \sum_{i=1}^{k} L_i \ , \tag{3.4}$$

and

$$D(x,t) = \sum_{i=1}^{k} D_i \ , \tag{3.5}$$

where $L(x,t)$ and $D(x,t)$ are the BOD and OD levels at location x and time t.

Suppose that immediately before t_1 there are background BOD and OD concentrations of L_o and D_o. The above analysis can be correspondingly modified by replacing c_1 by $c_1 + L_o$, and letting the initial OD level be D_o for the first application of Dobbins equation.

The value of k, the number of loads received by a particular segment of water, will depend upon the ration of fresh water velocity of maximum tidal velocity. If this ratio is small the segment will move out of range of the source slowly and k will be large. If the ratio is large the converse will be true. If there is no fresh water flow infinitely many roots exist for Equation (3.1). The magnitude of the component contribution by each load will depend upon two factors: the size of the load, and the reaction time for the load. Loads received a sufficient length of time in the past will result in a negligible effect on the total. Therefore, if k is large the calculations may be reduced by considering only roots of Equation (3.1) larger than some threshold value. The value of this threshold will depend upon the decay parameters k_1 and k_2.

Numerical Results for the Non-Dispersion Model

In all of the formulae derived above U(t) was an arbitrary function. For the following numerical examples U(t) consists of two components; (a) constant fresh water flow, (b) sinusoidal tidal velocity. Thus

$$U(t) = U_f + U_T \cos \omega t \quad , \qquad (3.6)$$

where

U_f is the fresh water velocity,

U_T is the amplitude of the tidal velocity, and

ω is the frequency, i.e., if the time is in hours and 12 hours represent a tidal cycle, then $\omega = \pi/6$.

If time is in hours and the tidal cycle is 12 hours,

0.0 hours represents positive rush water,

3.0 hours represents ebb tide,

6.0 hours represents negative rush water, and

9.0 hours represents flood tide.

With a periodic velocity function, the estuary is described by a pseudo-steady state condition. Pseudo-steady state implies that the concentration distributions are constant between tidal cycles but vary within a cycle. For example, the flood tide distributions will be constant for each cycle but may differ from the ebb tide distributions. Under these conditions, t in Equation (3.1) is the time of interest in the tidal cycle; and in general the t_i's (the roots of Equation (3.1)) will be negative.

Using U(t) as given in Equation (3.6) the integral in Equation (3.1) can be obtained. The roots can be found by appropriate numerical methods. In the ensuing program the iterative method of "False position" is used (31). A Fortran IV program was written to do this, and to perform the additional calculations necessary to evaluate L(x,t) and D(x,t). All of the programs alluded to in this chapter are shown in Appendix A.

For purposes of illustration, a hypothetical example was considered. The source emits pollution at a constant rate of 10,000 lbs. BOD/hr. The estuary has a cross-sectional area of 100,000 sq. ft., the fresh water velocity is one mile per day and the tidal velocity amplitude U_t is 25 miles per day. The deoxygenation and reaeration coefficients are 0.4 and 0.5, respectively, in per day units. The tidal cycle is 12 hours and background BOD and OD are negligible.

Figures 2 through 5 show the spatial profiles for BOD and oxygen deficit at different phases of the tidal cycle. These curves were obtained by fixing t and varying x in increments of 0.1 miles.

A second program varies t by increments of 0.1 hours, while fixing the distance from the source. The temporal profiles are shown in Figures 6 through 9 for several locations along the estuary.

It should be kept in mind that these curves represent the mean BOD and oxygen deficit levels. The wide variations of the concentration within the profiles are produced by the time dependent velocity. The probability distributions about the mean concentration will be discussed in the next section.

The explanation of the extreme peaks noted in Figures 2 through 9 is found by examining Equation (3.3). The velocity U(t) is zero for some values of t, so the segments of water which are at the source when U(t) is near zero receive a heavy load of pollution. One can see from Equation (3.3) that in theory these peaks become infinite; however, in the numerical results this is not possible. In the spatial profiles the distance between the peaks is the distance the water advances due to fresh water flow during one tidal cycle; for this example one-half mile. In the temporal profiles one would expect the curves to be symmetric about 3.0 and 9.0 hours. Slight asymmetry is observed since concentrations were evaluated at discrete intervals.

The concept of the concentrations reaching infinity is theoretically unsatisfactory and in practice absurd. A paradox arises when U(t) equals zero. According to Equation (3.3) the concentrations become infinite, however, the duration of the zero velocity is also zero. Thus there is no loading during this period. This paradox can be averted by specifying some small but finite segment length Δx. The terms L(x,t) and D(x,t) represent the concentrations in an interval of length Δx about the point x. When the segment passes the source, the time required for the elemental volume of length Δx to pass is noted. The time interval is denoted by Δt. Using these values of Δx and Δt is Equation (3.2) the added concentration can be found. With this approach the concentrations remain finite.

Fortran IV programs for the above approach appear in Appendix A. In Figures 10 through 17 Δx is 0.1 mile. Figures 10 through 13 represent the spatial profiles while 14 through 17 correspond to the temporal profile. The observed differences between these curves and the earlier curves are academic; however, this approach is theoretically more pleasing. The maximum concentration levels are lower, as would be expected, and the symmetry in the temporal profiles is more pronounced.

A third situation was considered. Suppose numerous sources are located in a short stretch of the estuary. It would be reasonable to consider the sum effect of these sources as a uniform line source for some distance along the longitudinal axis of the estuary. This has the effect of smoothing out the concentration curve. Figures 18 through 21 show the spatial profiles for a source length of 1 mile. All other parameters are unchanged. Figures 22 through 25 represent the temporal profiles for the same situation.

The line source approach may also be thought of as an artificial method of introducing dispersion. The pollution entering from a point source is instantaneously dispersed over a mixing length, corresponding to the length of the line source. No additional longitudinal mixing occurs. It is interesting to compare this approach with O'Connor's analysis. Figure 26 represents the spatial profile with the parameters as defined earlier and a mixing length of 4.0 miles; the length of the tidal excursion. Figure 27 represents the spatial profile derived by O'Connor for the same set of parameters and a dispersion coefficient of 5 mi.^{2}/day. Figure 28 is the temporal profile for the same situation as Figure 26. O'Connor's formulae do not lead to temporal variation; however, he admits that the concentrations vary within a tidal cycle. He discusses the temporal variations due to tides, and shows sketches which agree with Figure 28 (48).

Probability Distributions for Non-Dispersion Model

The analysis in the preceding section derived numerical values for the mean BOD and DO profiles by repeated applications of Dobbins' equations. It will be recalled that the effect of dispersion was neglected in favor of a varying velocity component. As a method of predicting mean BOD and DO levels it is doubtful that this model offers many advantages over the conventional methods. However, the rationale for the development was that a parallel stochastic approach existed which would yield complete BOD and DO distributions.

The principle of superposition implies that the distribution resulting from the individual pollution loadings is the convolution of the separate probability distributions. The probability generating function for each component is given by Equation (2.21), where t is the reaction time. Thus the distribution for the i^{th} load is obtained from $P(s,r,t-t_i)$. Let $P_{\tau}(s,r,t)$ be the generating function for the net distribution. Then

$$P_{\tau}(s,r,t) = \overset{k}{\underset{i=1}{\pi}} P(s,r,t-t_i) \quad . \qquad (3.7)$$

The initial conditions for each load are $L_o = L_i$, and $D_o = 0$. Utilizing Equation (2.21) one obtains

$$P_{\tau}(s,r,t) = \overset{k}{\underset{i=1}{\pi}} \left\{ \exp\ (1-\tau) \left[\frac{-\gamma\beta_1 i}{K_1+K_3} + \frac{\gamma\beta_3 i}{K_2} + \frac{D_B\beta_3\, i}{\Delta K_2} \right] \right.$$

$$\left. \frac{L_a\beta_1\, i}{\Delta(K_1+K_3)} (1-s) \right\} \left[1-(1-r)\ d_{2\,i} - (1-s)\ a_{1\,i} \right]^{L_i/\Delta} , \qquad (3.8)$$

where $a_1\, i = a_1\ (t-t_i)$, $\beta_1\, i = \beta_1\ (t-t_i)$, etc.

Letting

$$\beta_{1\tau} = \sum_{i=1}^{k} \beta_{1\,i}$$

$$\beta_{2\tau} = \sum_{i=1}^{k} \beta_{2i} ,$$

$$\beta_{3\tau} = \sum_{i=1}^{k} \beta_{3i} ,$$

Equation (3.8) becomes

$$P_{\tau}(s,r,t) = \exp \left\{ (1-r) \left[\frac{-\gamma\beta_{1\tau}}{K_1+K_3} + \frac{\gamma\beta_{3\tau}}{K_2} + \frac{D_B\beta_{2\tau}}{\Delta K_2'} \right] - \frac{L_a\beta_{1\tau}}{\Delta(K_1+K_3)} (1-s) \right\} \prod_{i=1}^{k} \left[1-(1-r)\,a_{2i} - (1-s)\,a_{1i} \right]^{L_i/\Delta} . \qquad (3.9)$$

The marginal generating function for pollution is obtained by setting r = 1, which yields,

$$P_{\tau}(s,1,t) = \exp \left[\frac{-L_a\beta_{1\tau}}{\Delta(K_1+K_3)} (1-s) \right] \prod_{i=1}^{k} \left[1-(1-s)\,a_{1i} \right]^{Li/\Delta} \qquad (3.10)$$

The distribution represented by $P_{\tau}(s,1,t)$ is equivalent to a distribution obtained from the convolution of two independent distributions. The first is represented by the generating function

$$P_1(s,1,t) = \exp\left[\frac{-L_a \beta_1 \tau}{\Delta(K_1+K_3)}(1-s)\right], \qquad (3.11)$$

which is the generating function of a Poisson Distribution with parameter

$$\frac{L_a \beta_1 \tau}{\Delta(K_1+K_3)}$$

The second distrubution possesses generating function

$$P_2(s,1,t) = \prod_{i=1}^{k} \left[1-(1-s)\,a_{1i}\right]^{L_i/\Delta} \qquad (3.12)$$

This distribution corresponds to $L_\tau \sum_{i=1}^{k} L_i/\Delta$ independent Bernoulli trails. The first L_1/Δ trails having probability of success a_{11}, the next L_2/Δ trails having probability of success a_{12}, and so forth. The coefficients of the individual powers of s may be found by performing the binomial multiplications. However, L_τ, the number of multiplications required, is usually large making this approach impractical. Fortunately a limiting approximation is readily available. By application of the central limit theorem the distribution represented by Equation (3.12) approaches a normal distribution with mean

$$\mu = \sum_{i=1}^{k} a_{1i} L_i/\Delta ,$$

and variance

$$\sigma^2 = \sum_{i=1}^{k} \alpha_{1i}\beta_{1i} \; L_i/\Delta \qquad ,$$

as

$$\sum_{i=1}^{k} L_i/\Delta \rightarrow \infty$$

Therefore,

$$P_2(s,1,t) \approx \sum_{i=1}^{k} a_i s^i \tag{3.13}$$

where a_i is obtained from the proper normal distribution.

Rewriting Equation (3.10) one obtains

$$P_\tau(s,1,t) \approx \exp\left[\frac{-L_a \beta_1 \tau}{\Delta(K_1+K_3)} (1-s) \right] \sum_{i=1}^{L_\tau} a_i s^i$$

$$= \exp\left[\frac{-L_a \beta_1 \tau}{\Delta(K_1+K_3)} \right] \sum_{j=0}^{\infty} \left[\frac{L_a \beta_1 \tau s}{\Delta(K_1+K_3)} \right]^j \frac{1}{j!} \sum_{i=1}^{L_\tau} a_i s^i \; . \tag{3.14}$$

If p_m represents the net probability of being in BOD state m, one has

$$p_m = \exp \left[\frac{-L_a \beta_1 \tau}{\Delta(K_1+K_3)} \right] \sum_{j=0}^{m^*} a_j \left[\frac{L_a \beta_1 \tau}{\Delta(K_1+K_3)} \right]^{m-j} \frac{1}{(m-j)!} ,$$

where

$$m^* = \begin{cases} m \text{ for } m < L_\tau \\ L_\tau \text{ for } m \geqslant L_\tau . \end{cases} \tag{3.15}$$

A similar argument concerning the marginal distribution for the oxygen deficit state yields

$$p_n = \exp \left[\frac{-\gamma \beta_1 \tau}{K_1+K_3} + \frac{\gamma \beta_3 \tau}{K_2} + \frac{D_B \beta_3 \tau}{\Delta K_2} \right]$$

$$\sum_{j=0}^{n^*} b_j \left[\frac{\gamma \beta_1 \tau}{K_1+K_3} - \frac{\gamma \beta_3 \tau}{K_2} - \frac{D_B \beta_3 \tau}{\Delta K_2} \right]^{n-j} \frac{1}{(n-j)!} ,$$

where

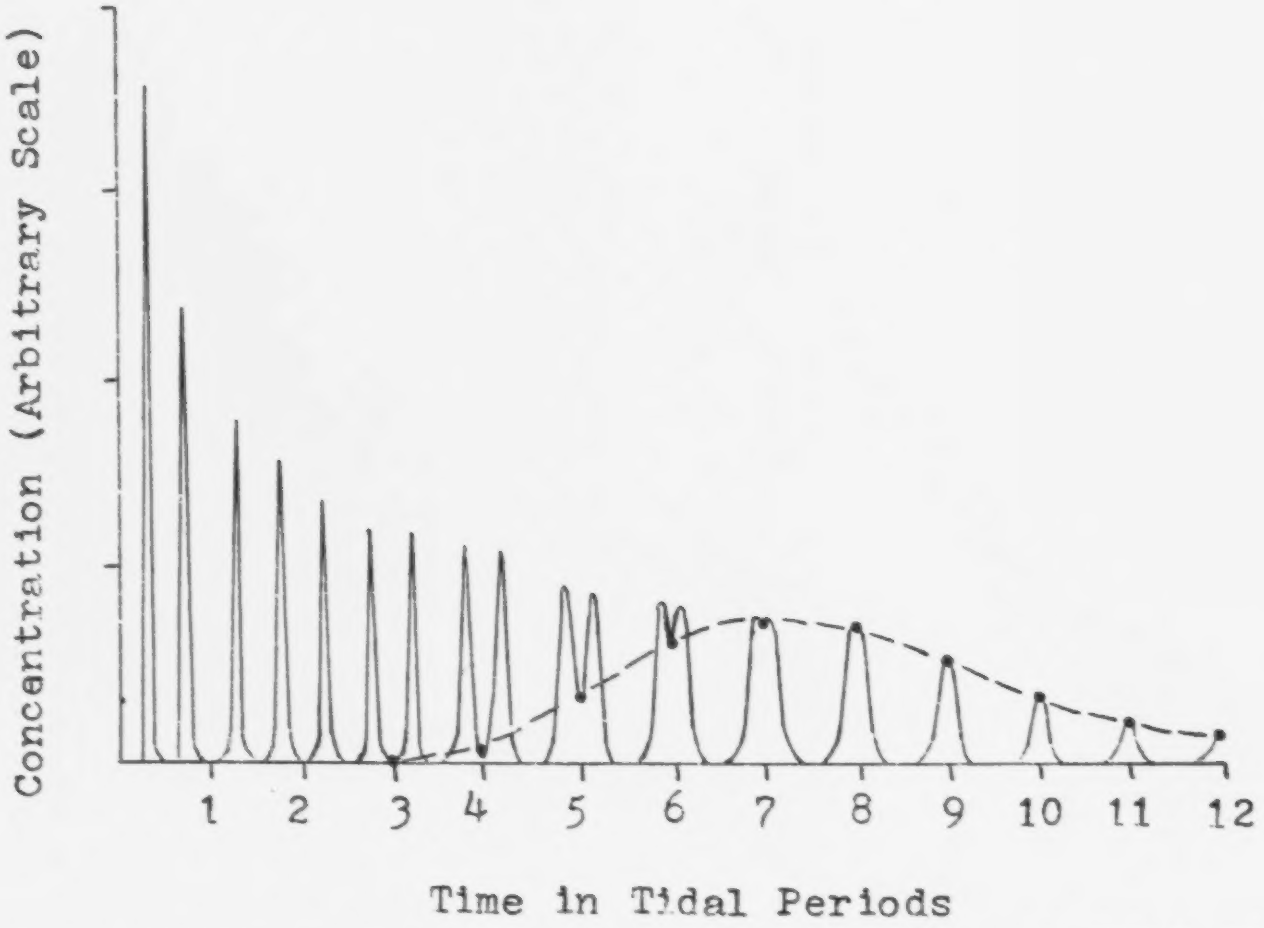
$$n^* = \begin{cases} n \text{ for } n < L_\tau \\ L_\tau \text{ for } n \geqslant L_\tau \end{cases} , \qquad (3.16)$$

where p_n represents the probability of the OD being in state n, and b_n is obtained from a normal distrubution with mean

$$\sum_{i=1}^{k} a_{2i} L_i / \Delta ,$$

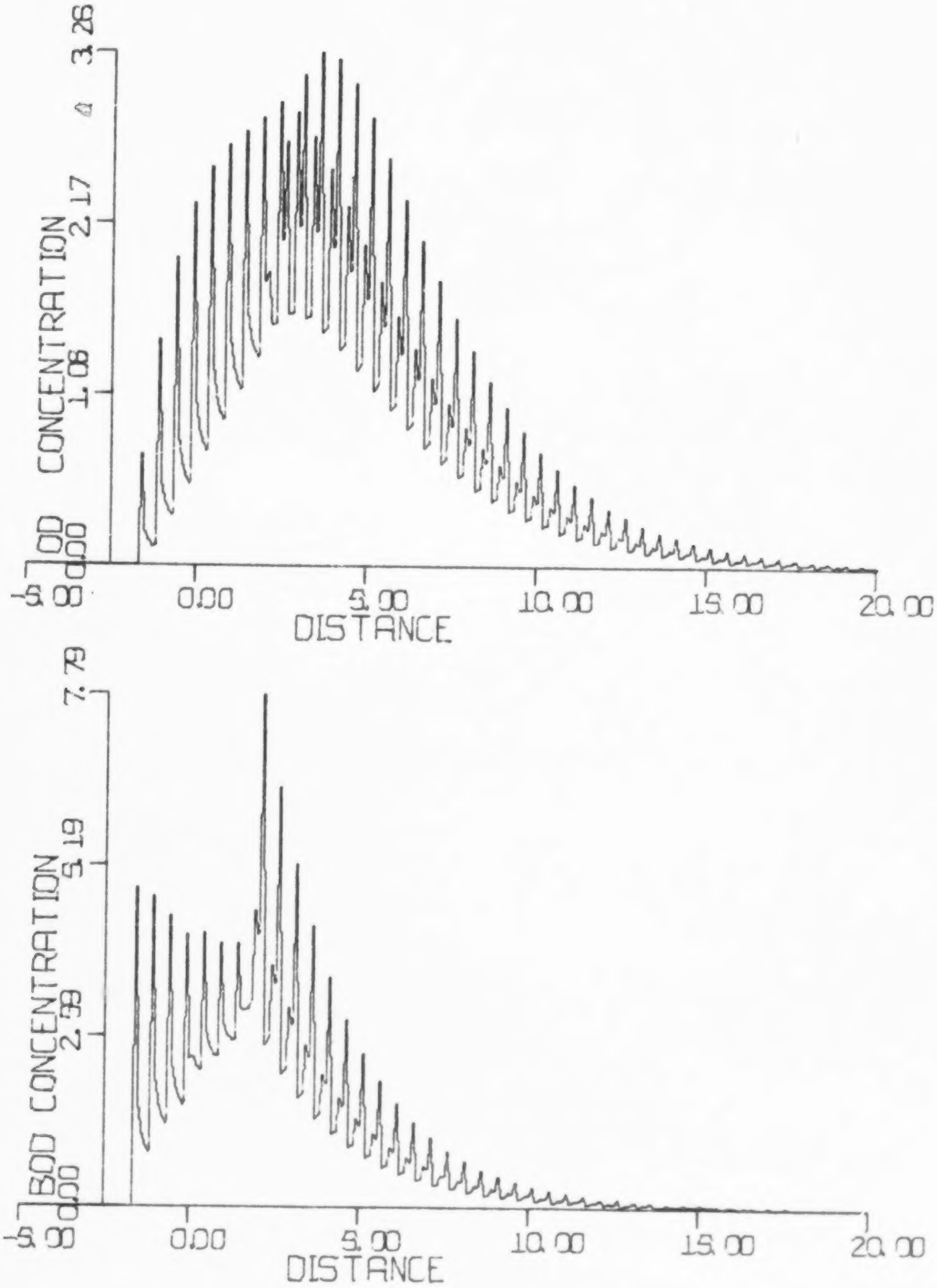
and variance

$$\sum_{i=1}^{k} a_{2i} \beta_{2i} L_i / \Delta$$



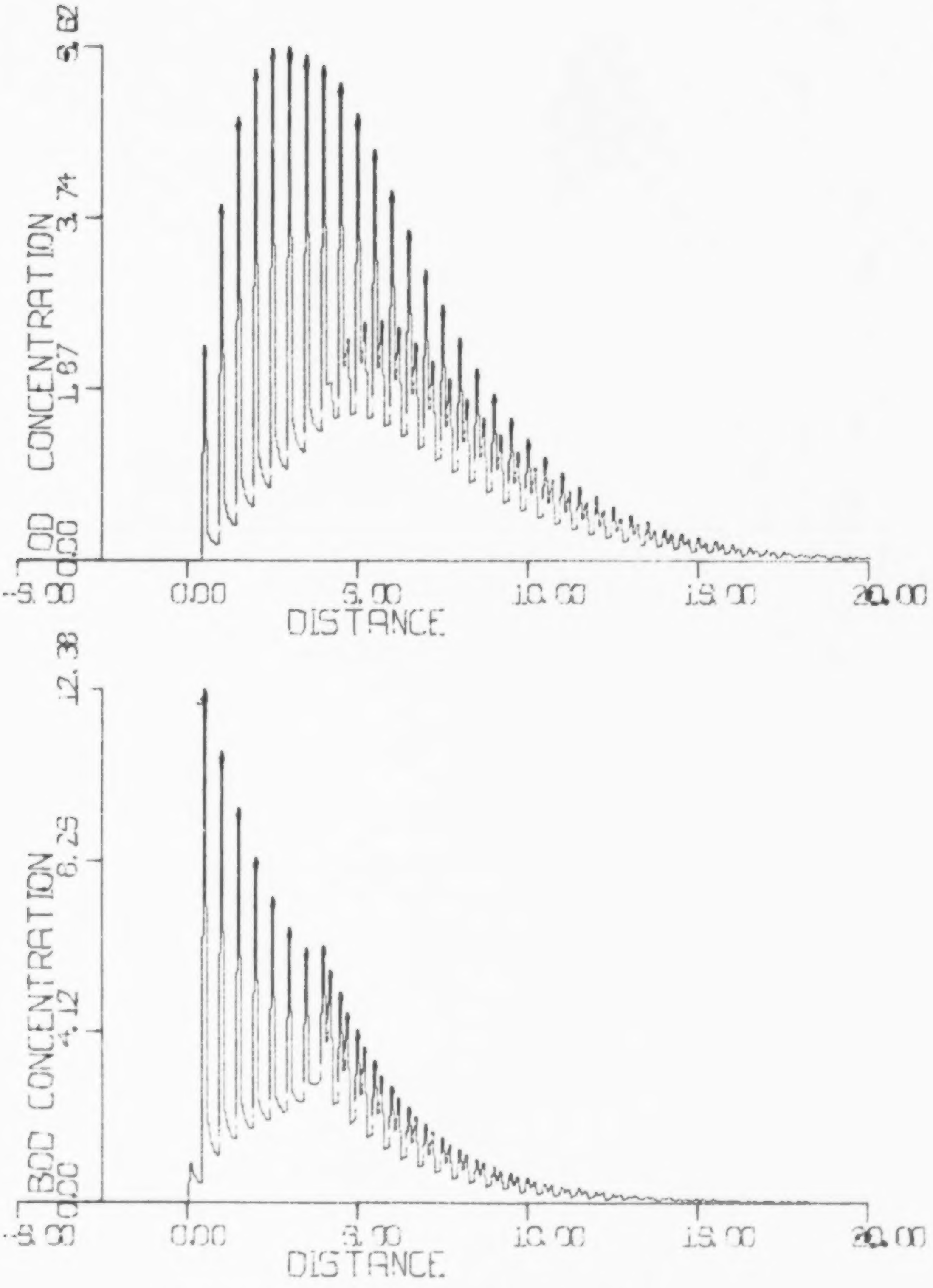
THEORETICAL CONCENTRATION DISTRIBUTION IN ESTUARY-TYPE FLOW FOR INSTANTANEOUS, POINT INJECTION

FIGURE 1



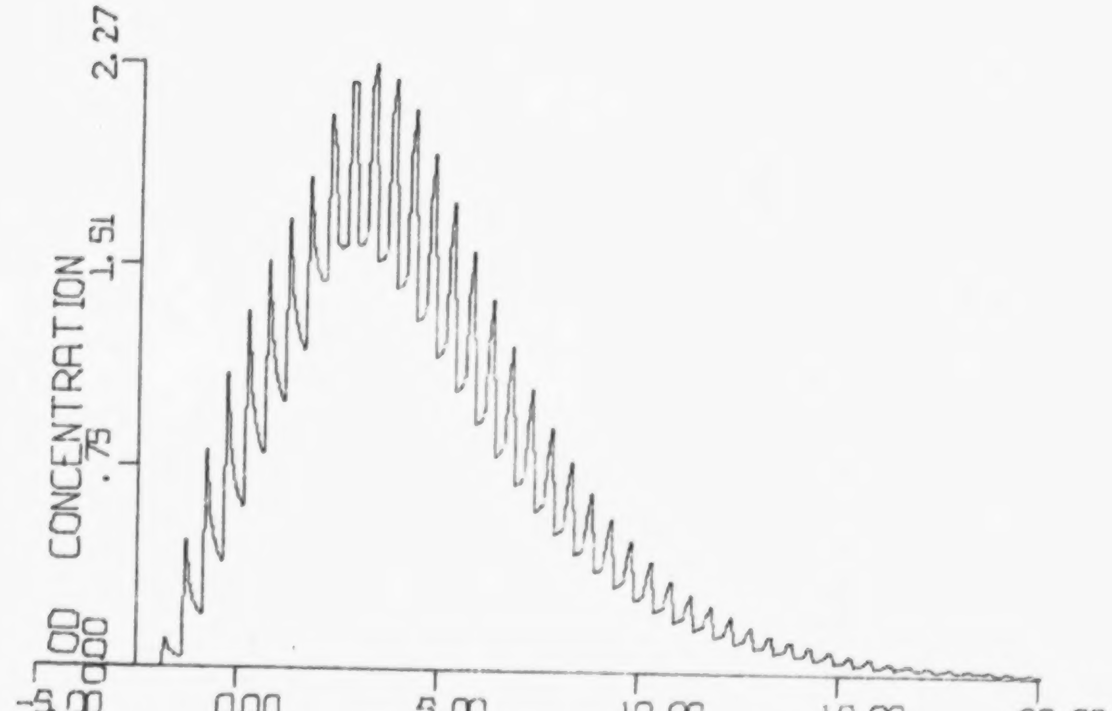
NON-DISPERSION MODEL - POINT SOURCE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 2



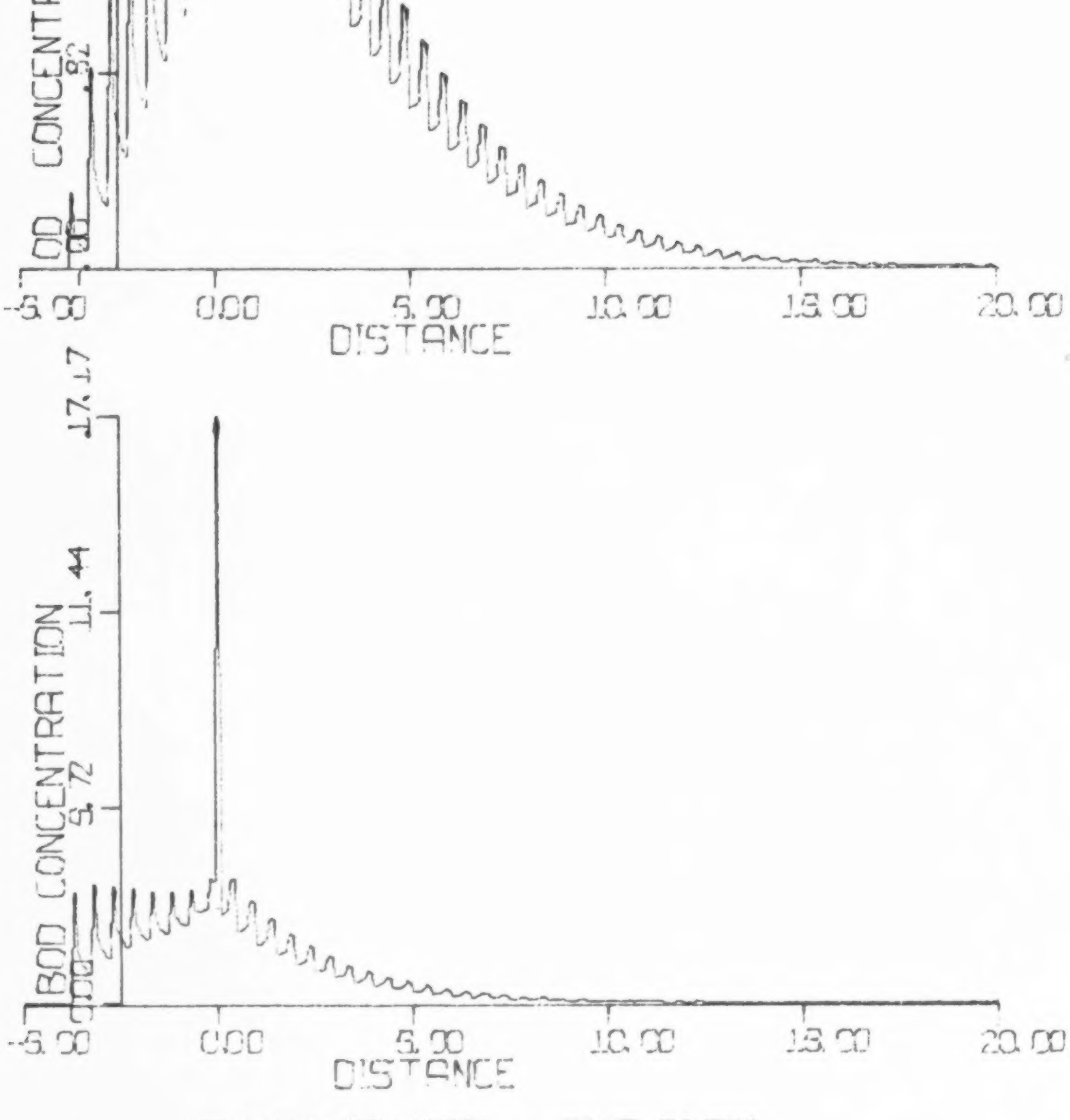
NON-DISPERSION MODEL - POINT SOURCE
B.O.D. AND O.D. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE

FIGURE 3



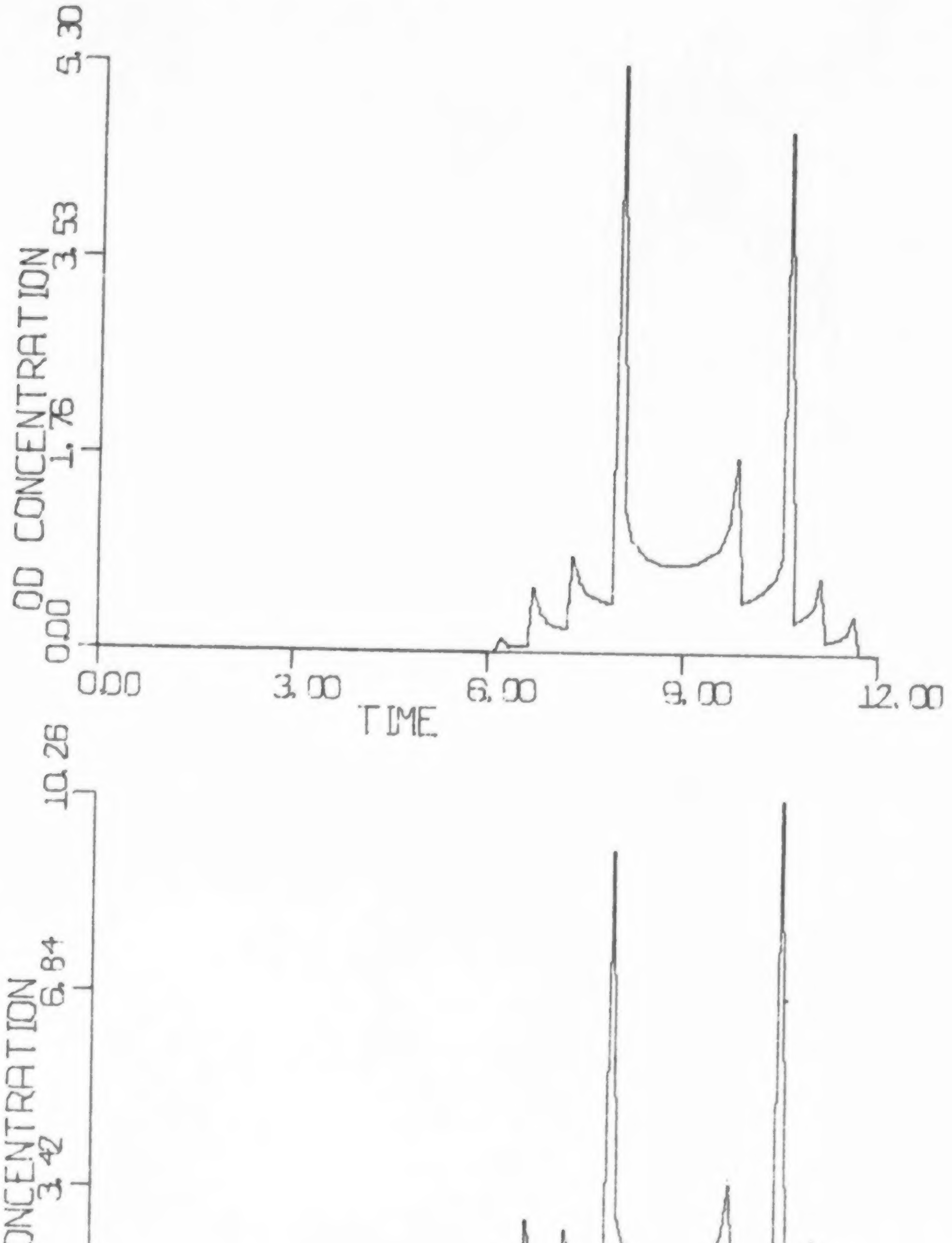
NON-DISPERSION MODEL - POINT SOURCE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 4



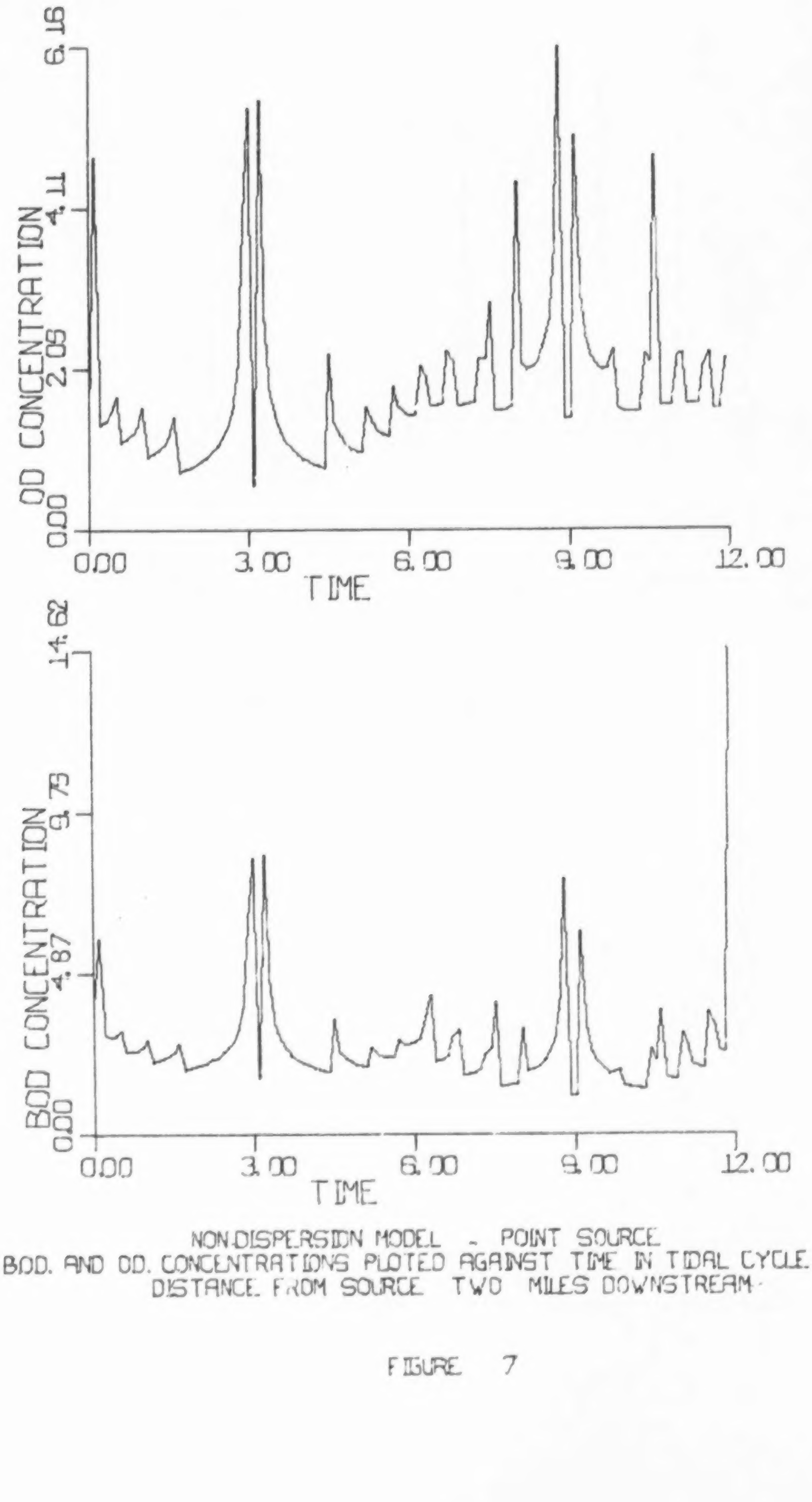
NON-DISPERSION MODEL - POINT SOURCE
B.O.D. AND O.D. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE

FIGURE 5



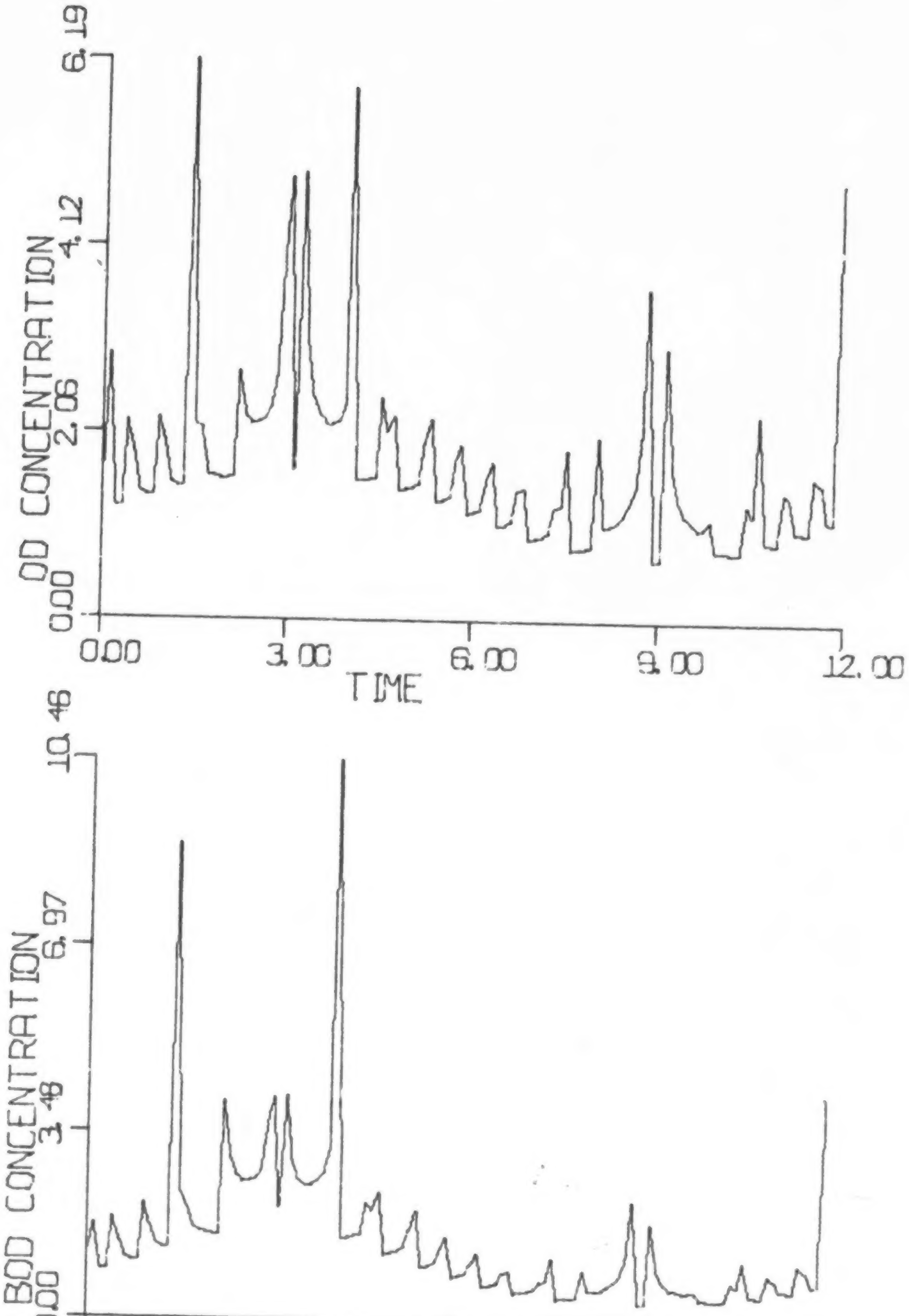
NON-DISPERSION MODEL - POINT SOURCE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 6



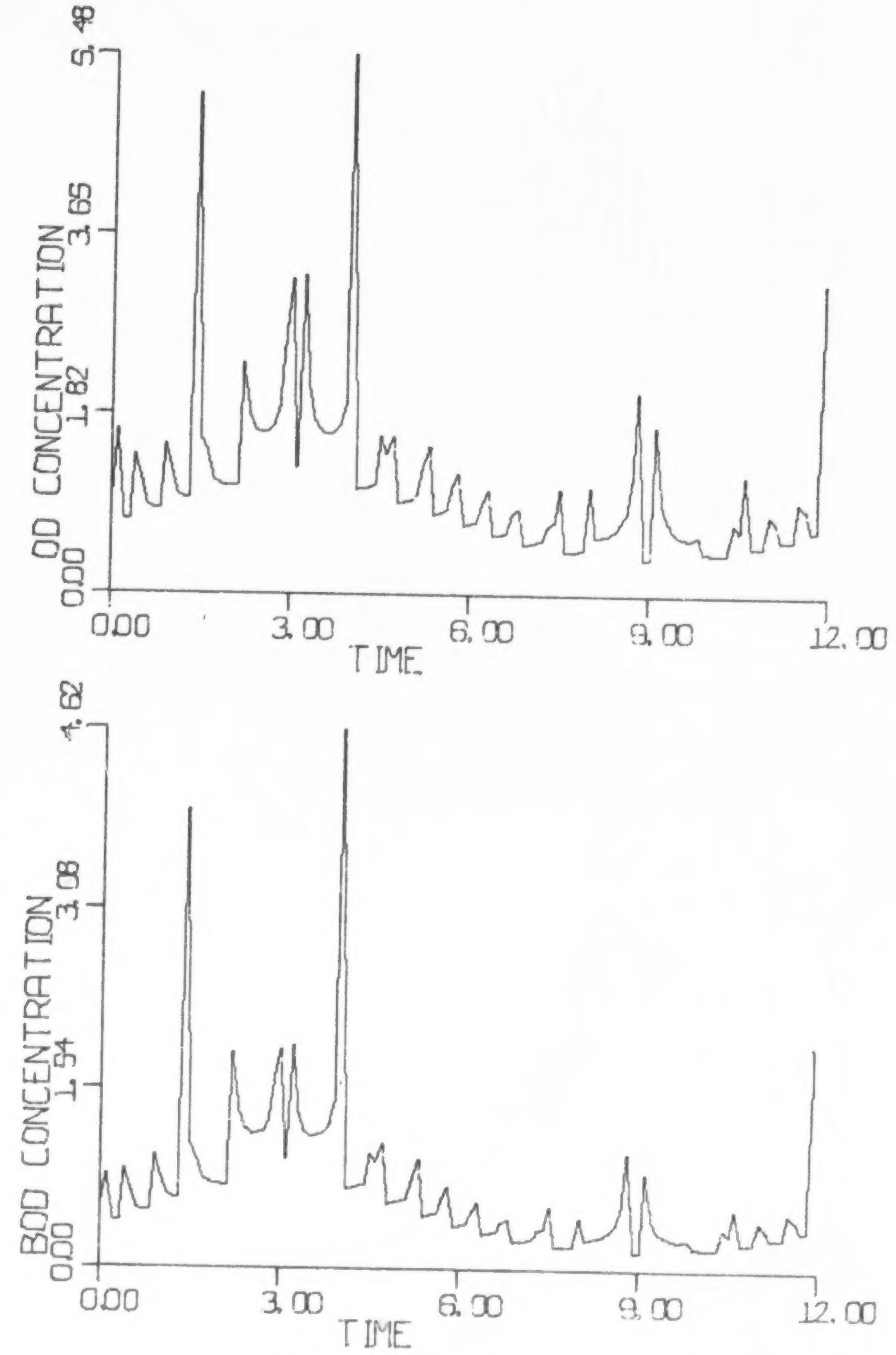
NON-DISPERSION MODEL - POINT SOURCE
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM.

FIGURE 7



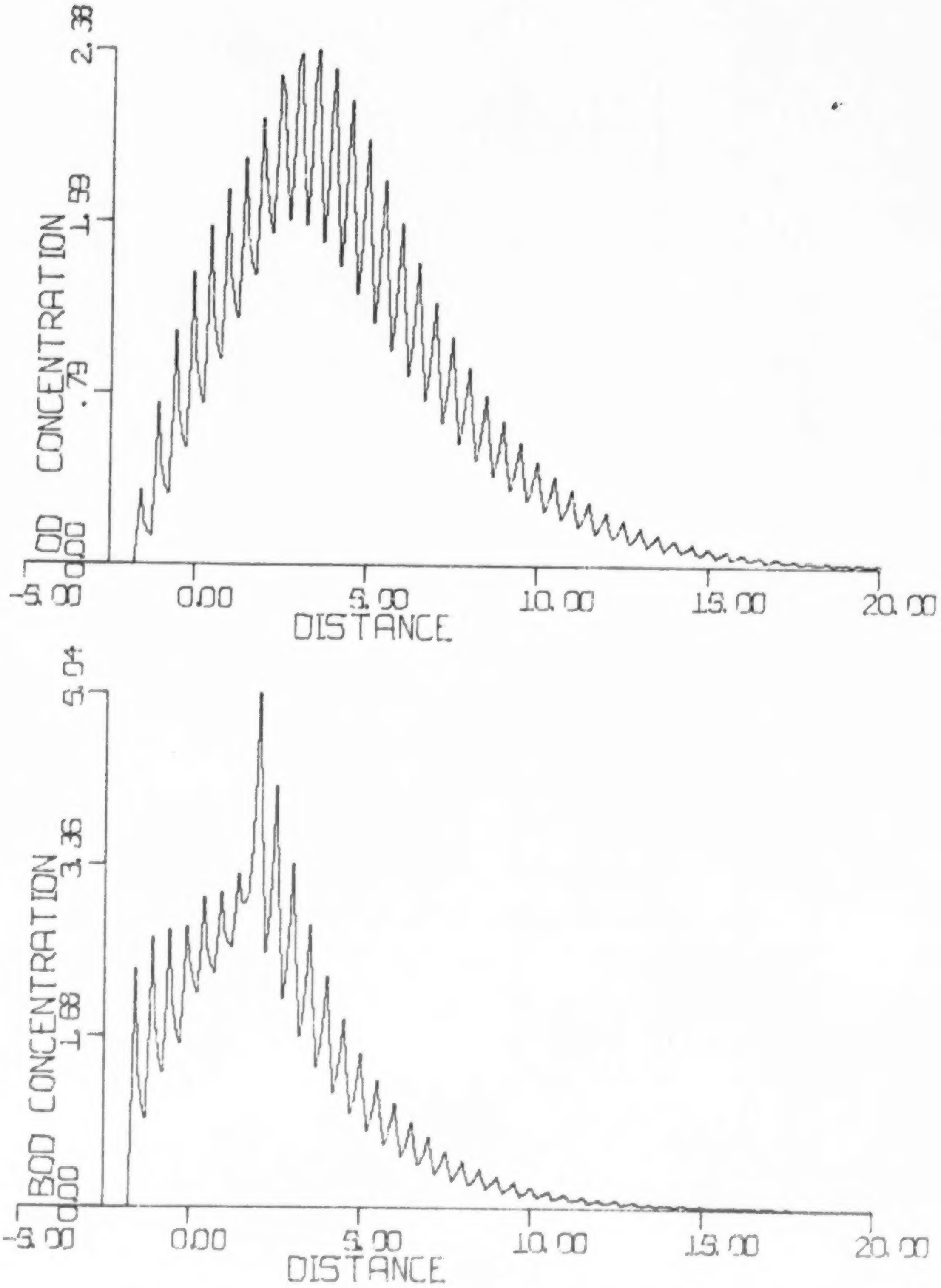
NON-DISPERSION MODEL - POINT SOURCE
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 8



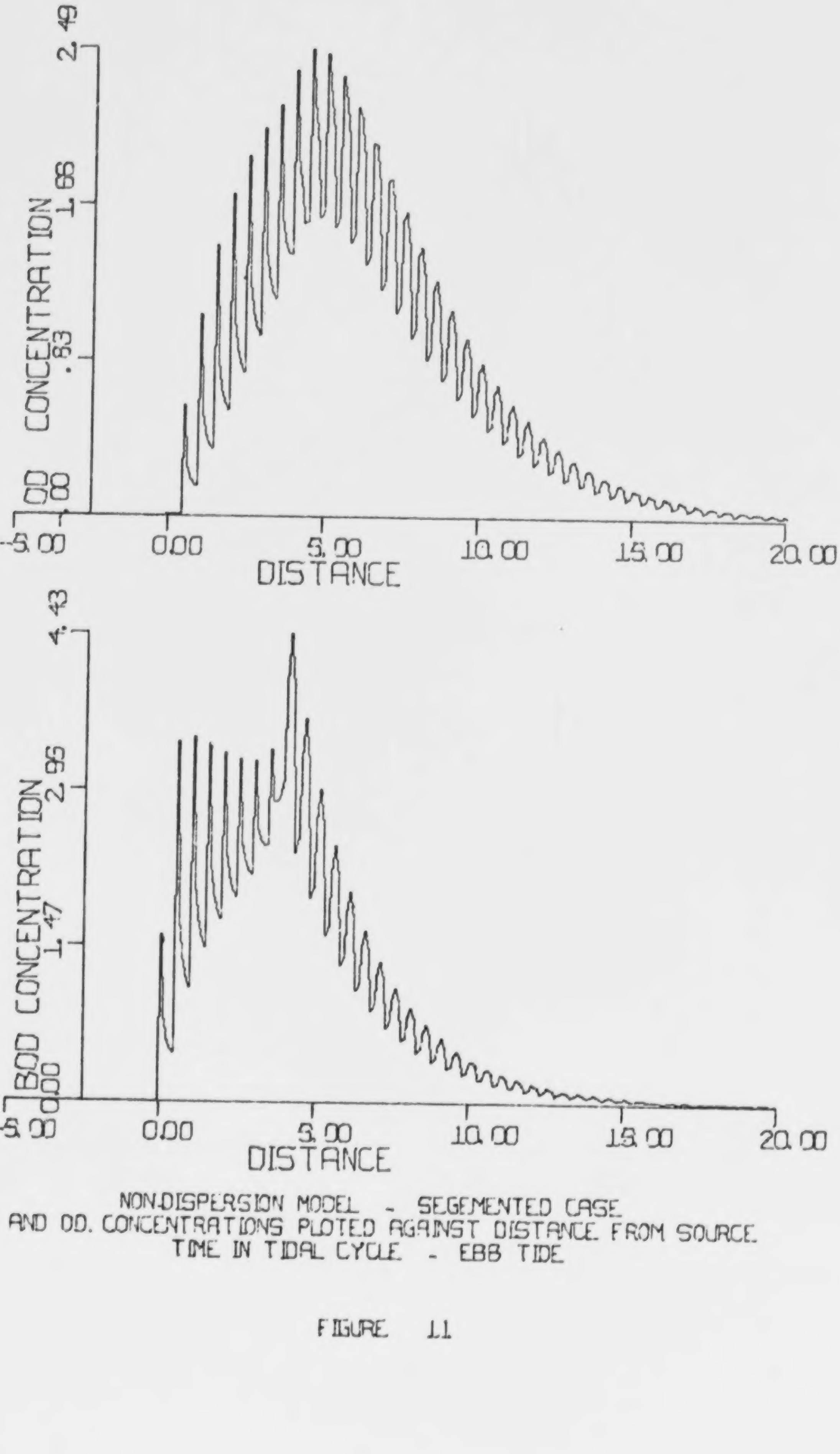
NON-DISPERSION MODEL - POINT SOURCE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES UPSTREAM

FIGURE 9



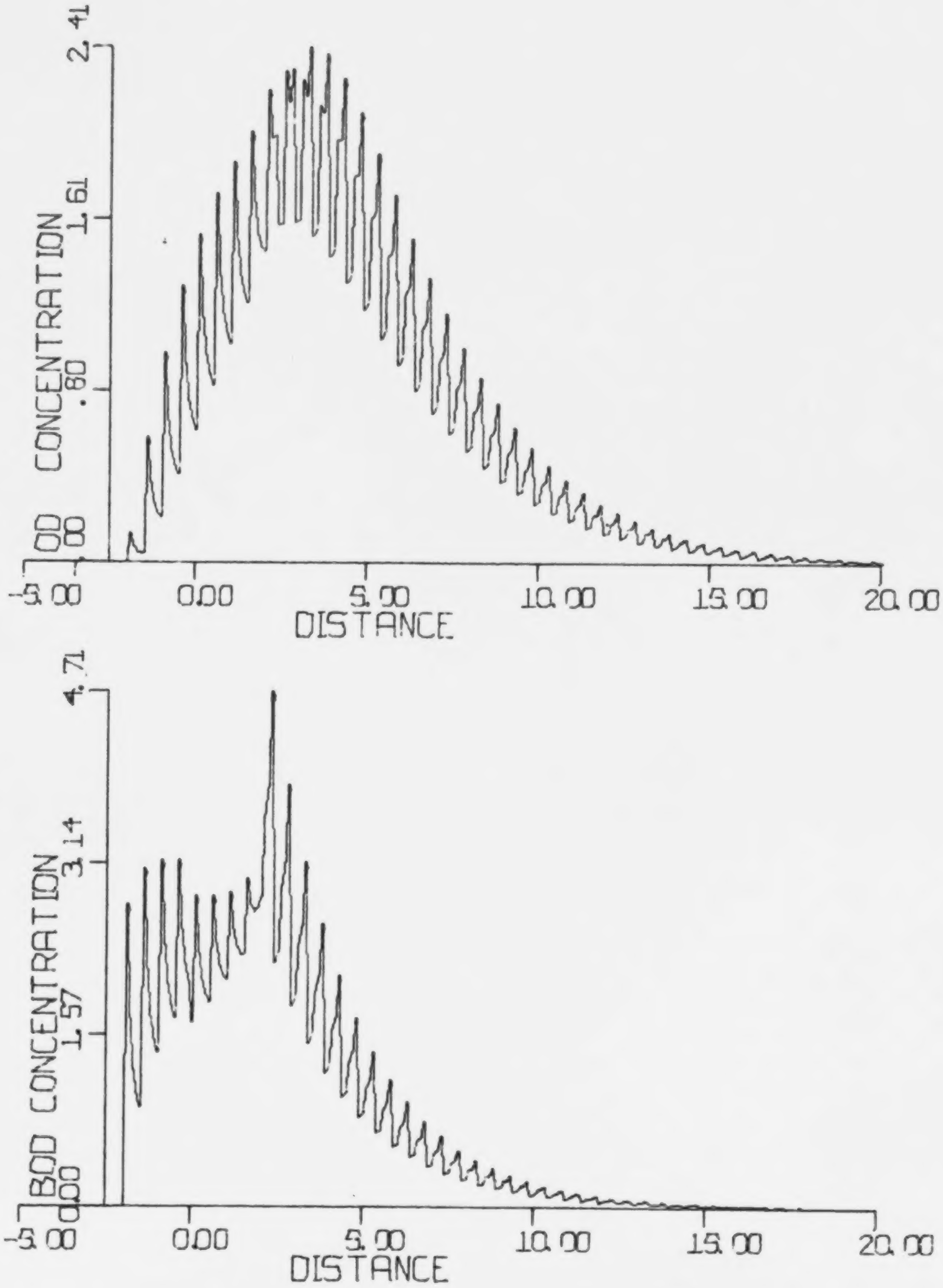
NON-DISPERSION MODEL - SEGEMENTED CASE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 10



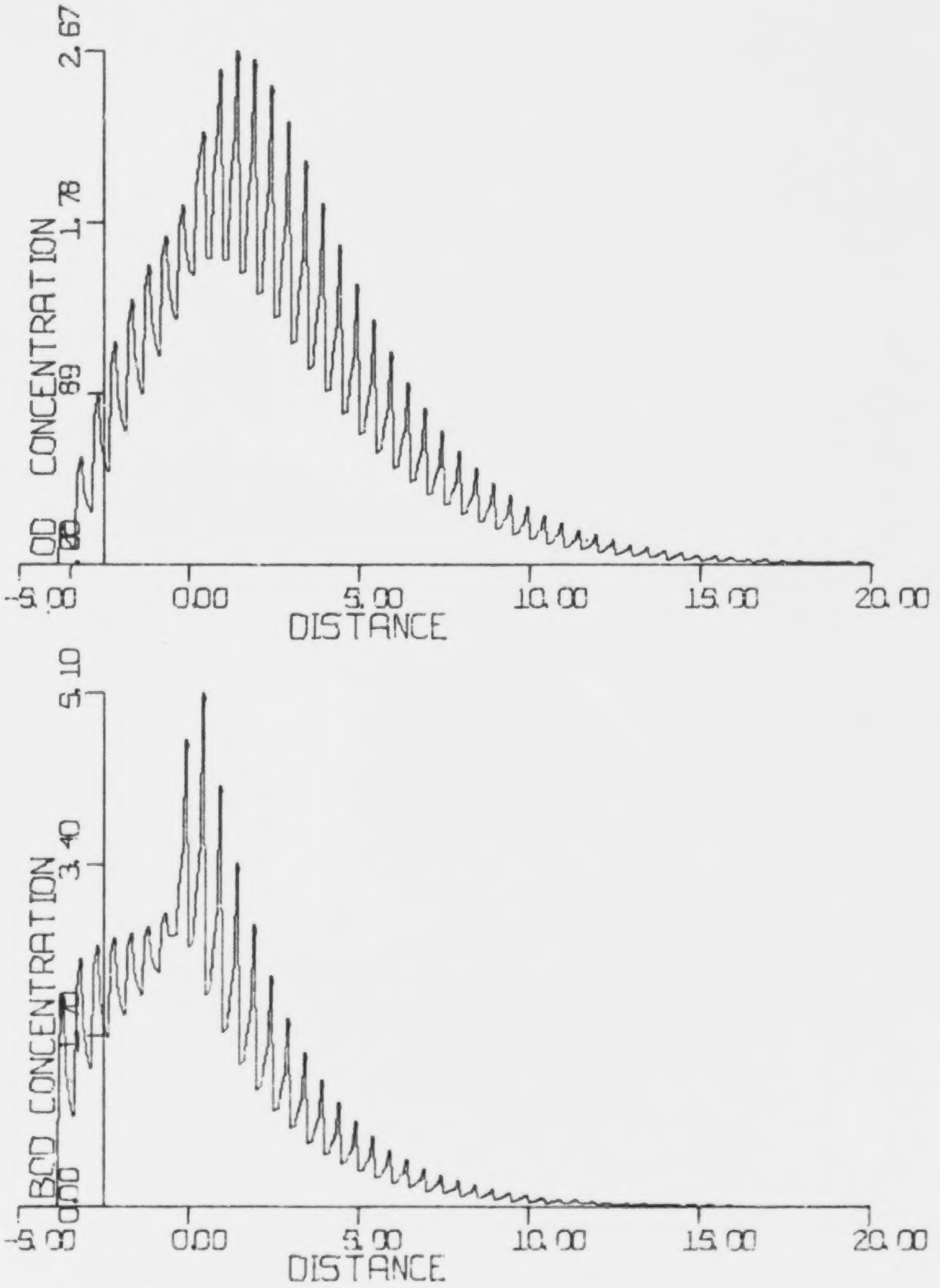
NON.DISPERSION MODEL - SEGEMENTED CASE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE

FIGURE 11



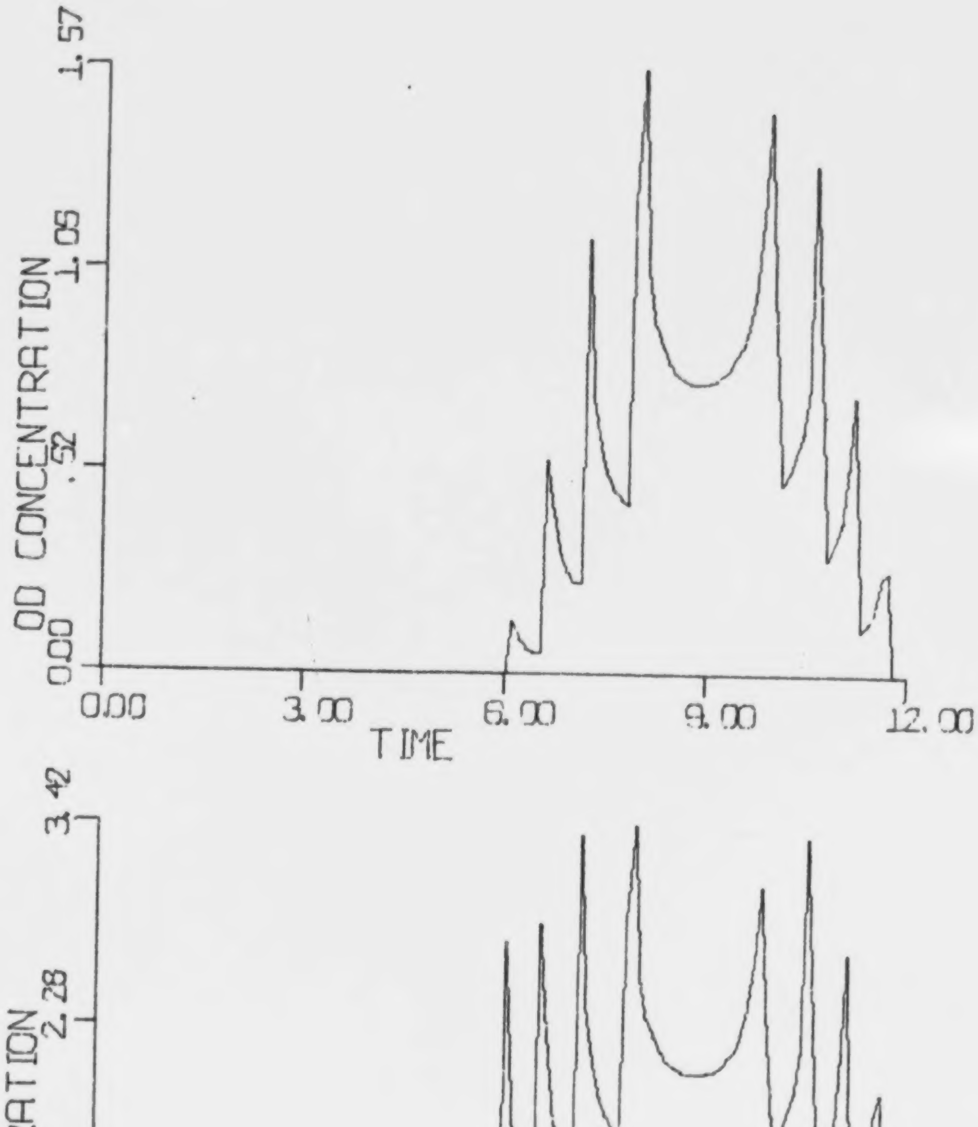
NON.DISPERSION MODEL - SEGEMENTED CASE
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 12



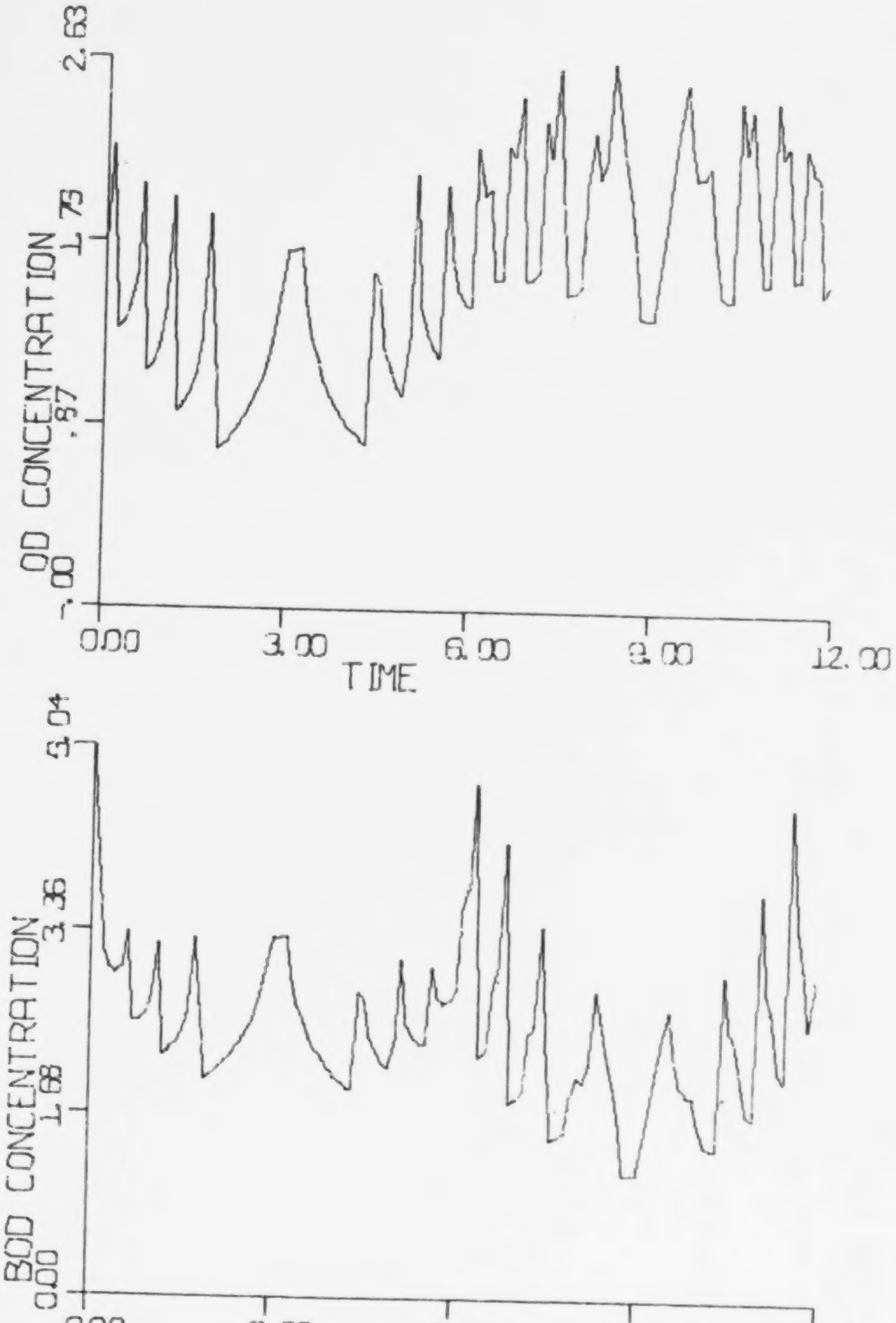
NON-DISPERSION MODEL - SEGEMENTED CASE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE

FIGURE 13



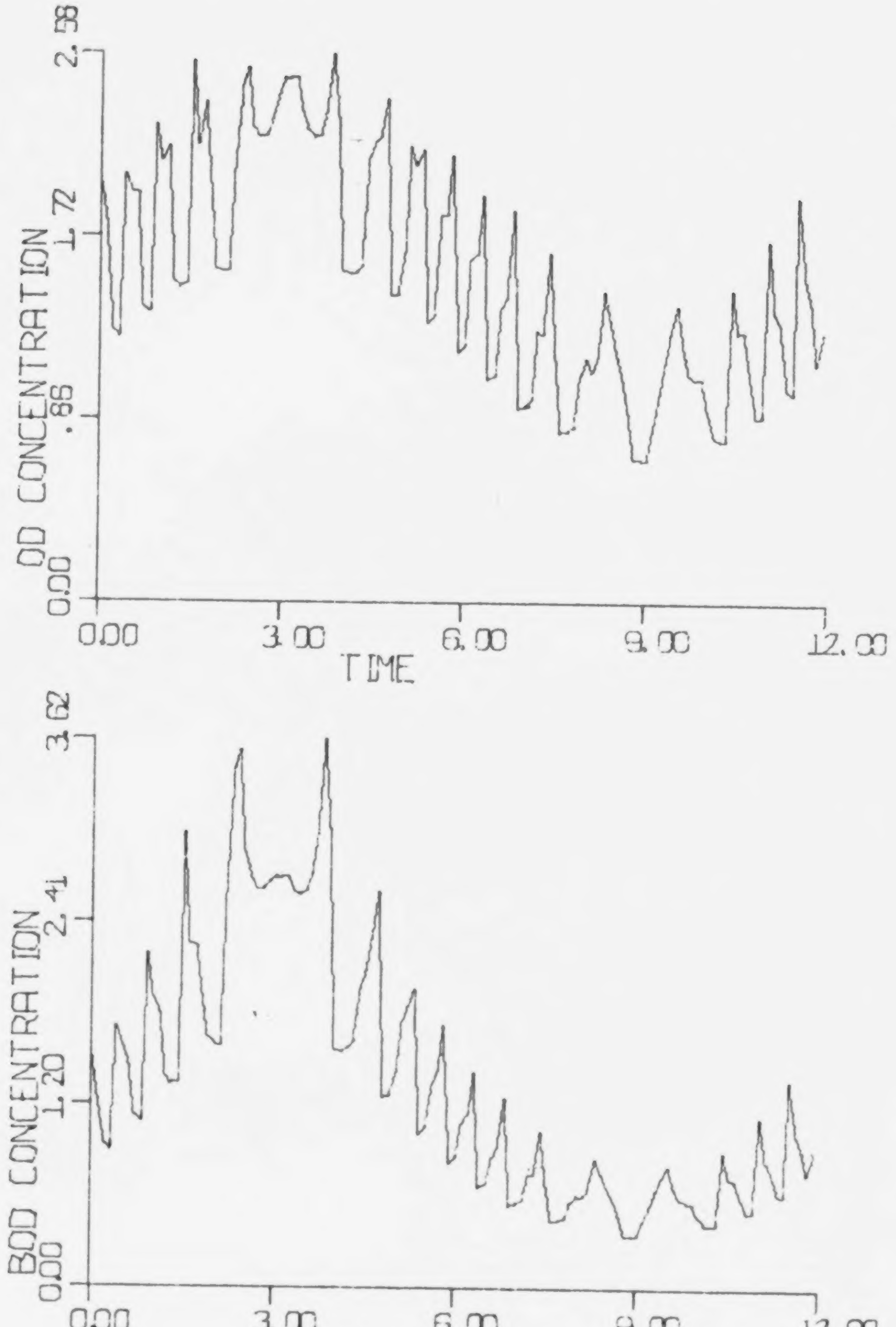
NON-DISPERSION MODEL - SEGEMENTED CASE
B.O.D. AND O.D. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 14



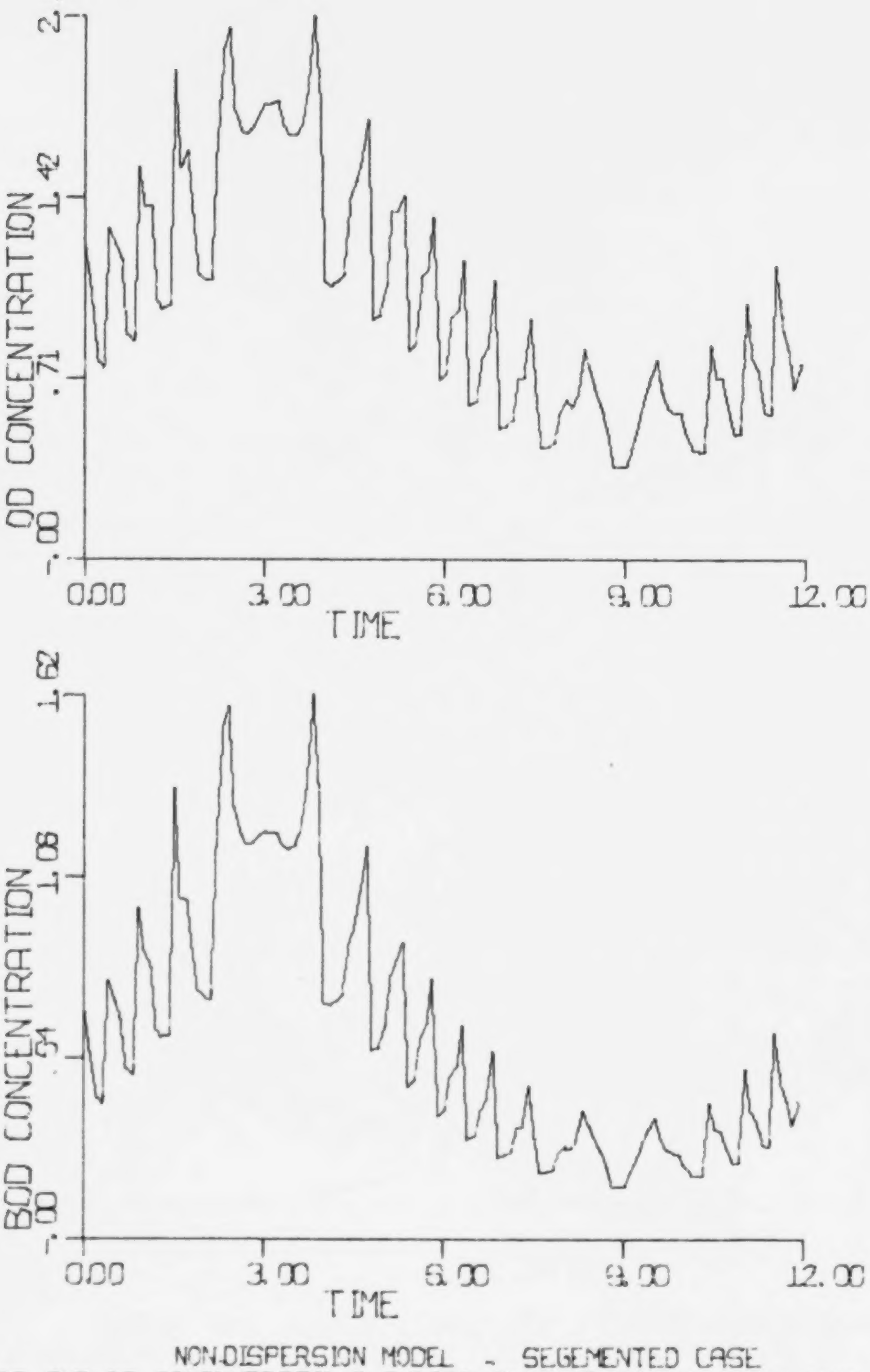
NON-DISPERSION MODEL - SEGEMENTED CASE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM

FIGURE 15



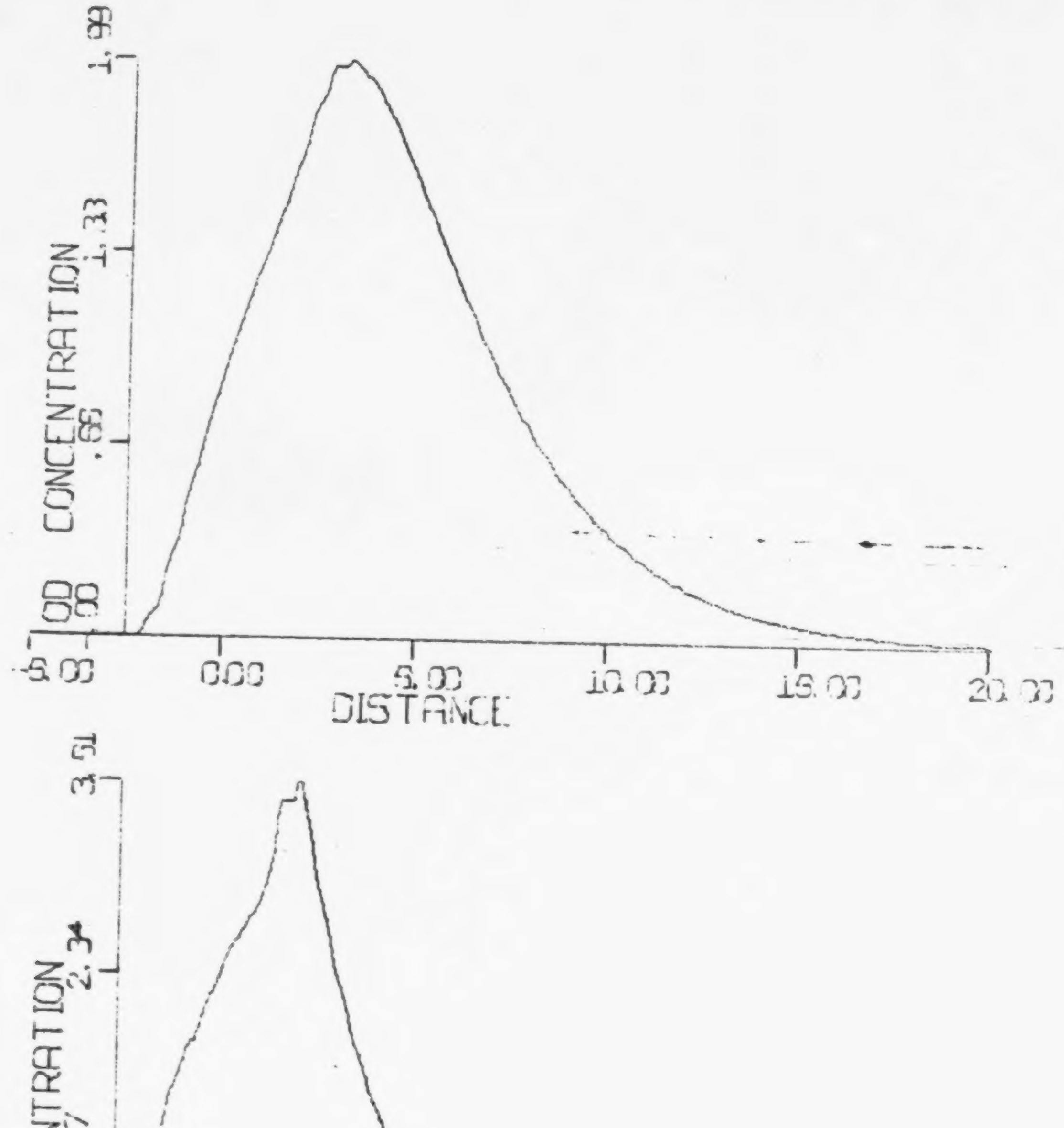
NON-DISPERSION MODEL - SEGEMENTED CASE
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 16



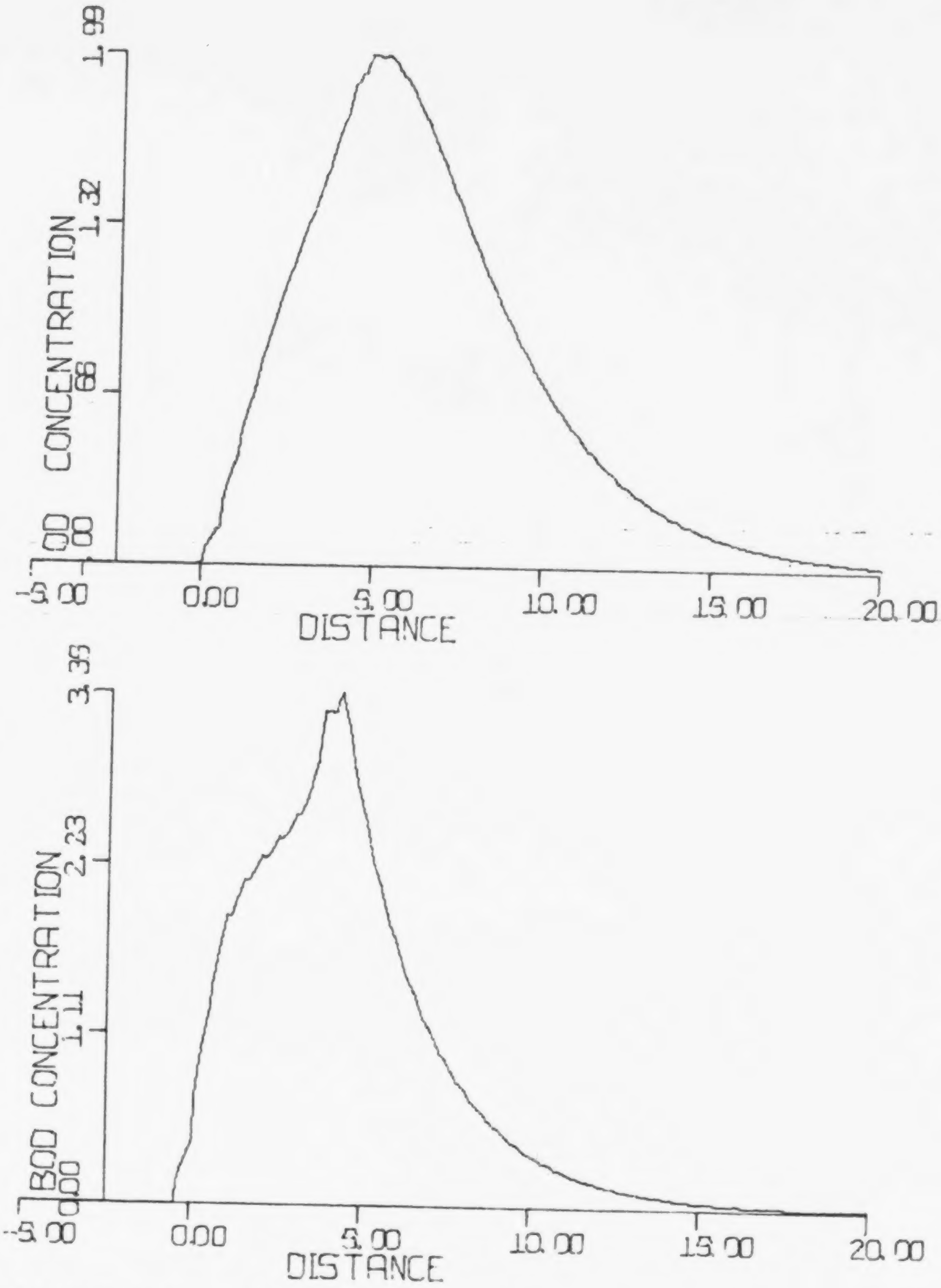
NON-DISPERSION MODEL - SEGEMENTED CASE
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 17



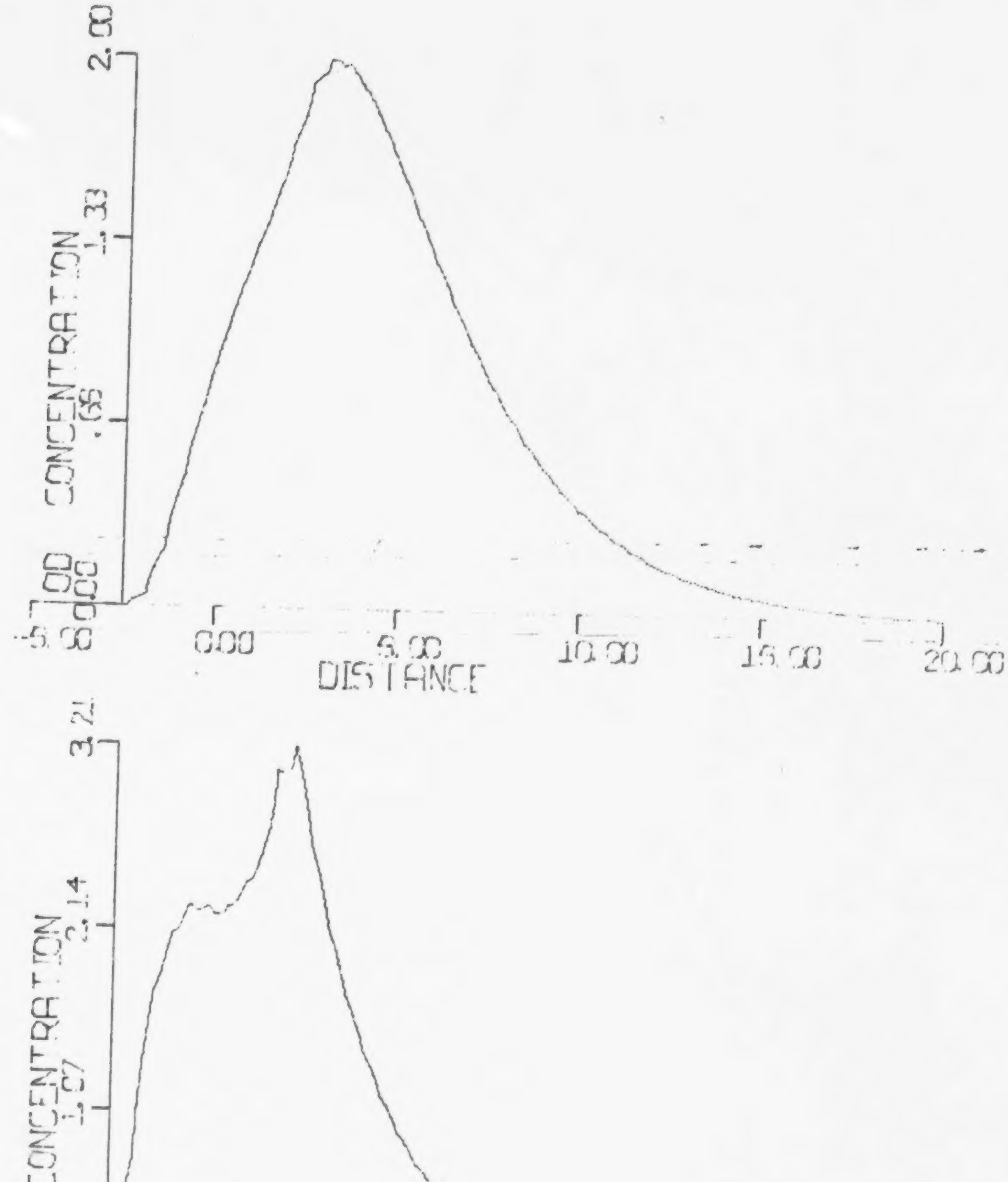
NON-DISPERSION MODEL - LINE SOURCE, LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 16



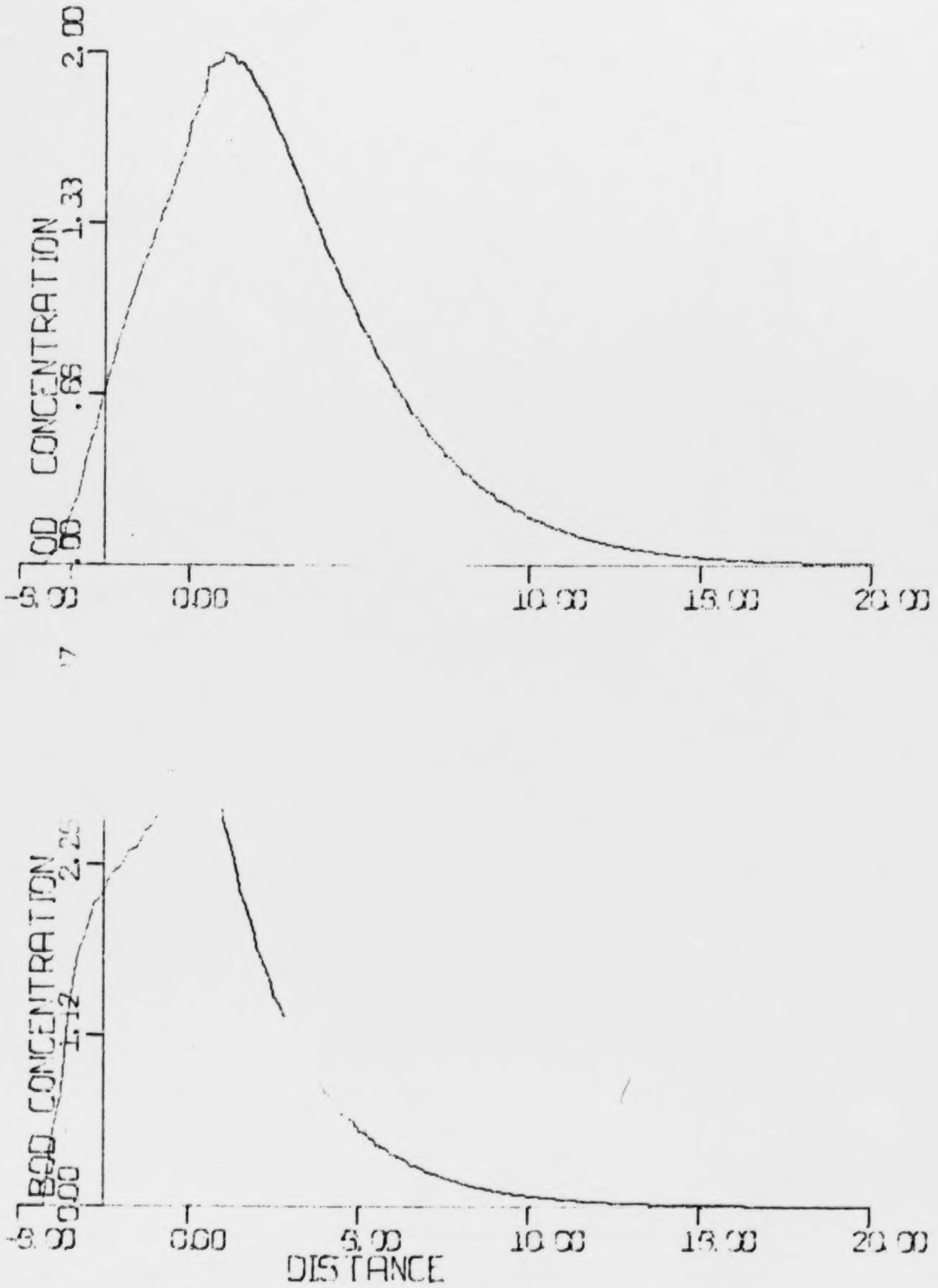
NON-DISPERSION MODEL - LINE SOURCE LENGTH ONE MILES
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE.

FIGURE 19



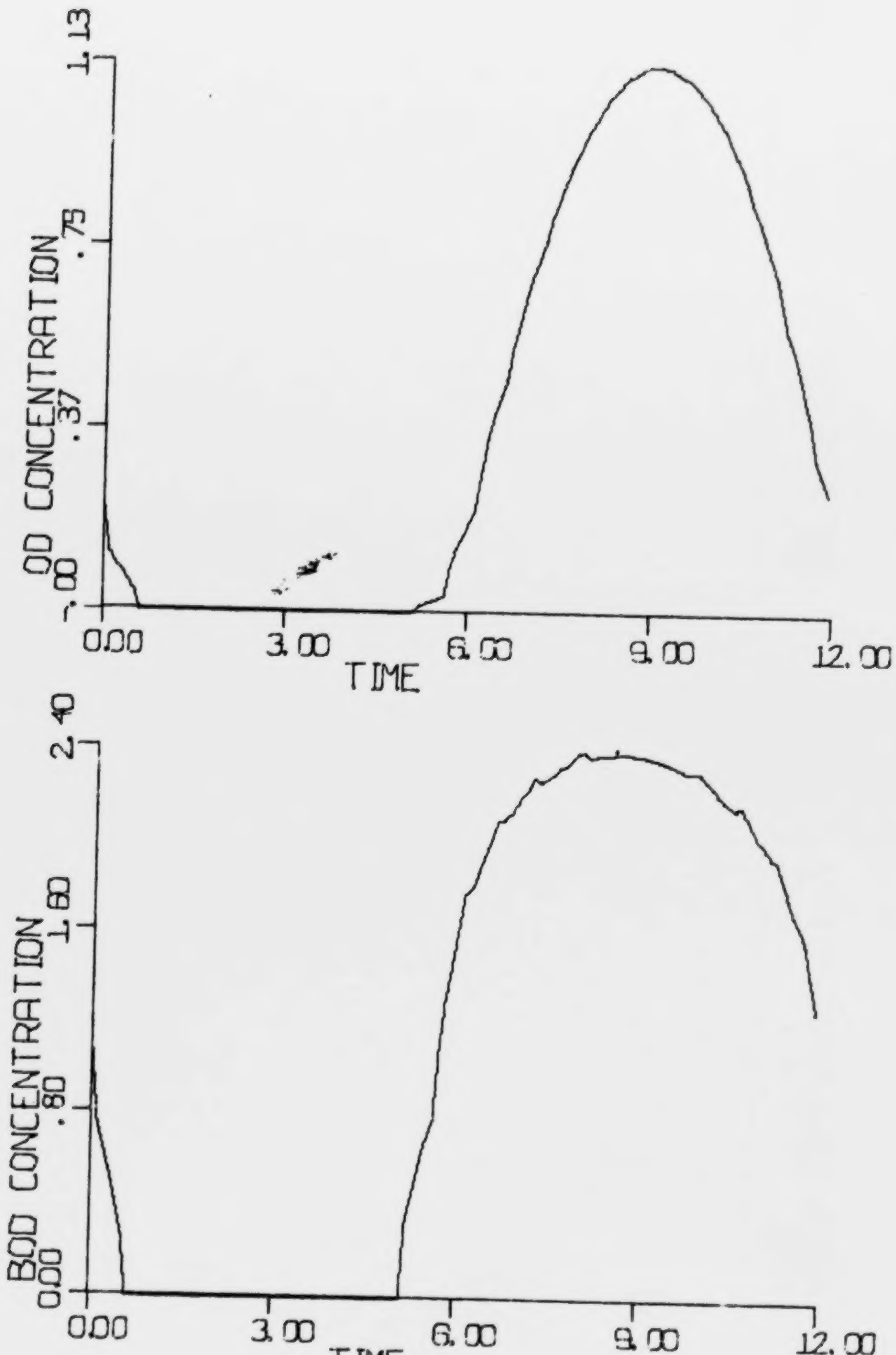
NON-DISPERSION MODEL - LINE SOURCE, LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 20



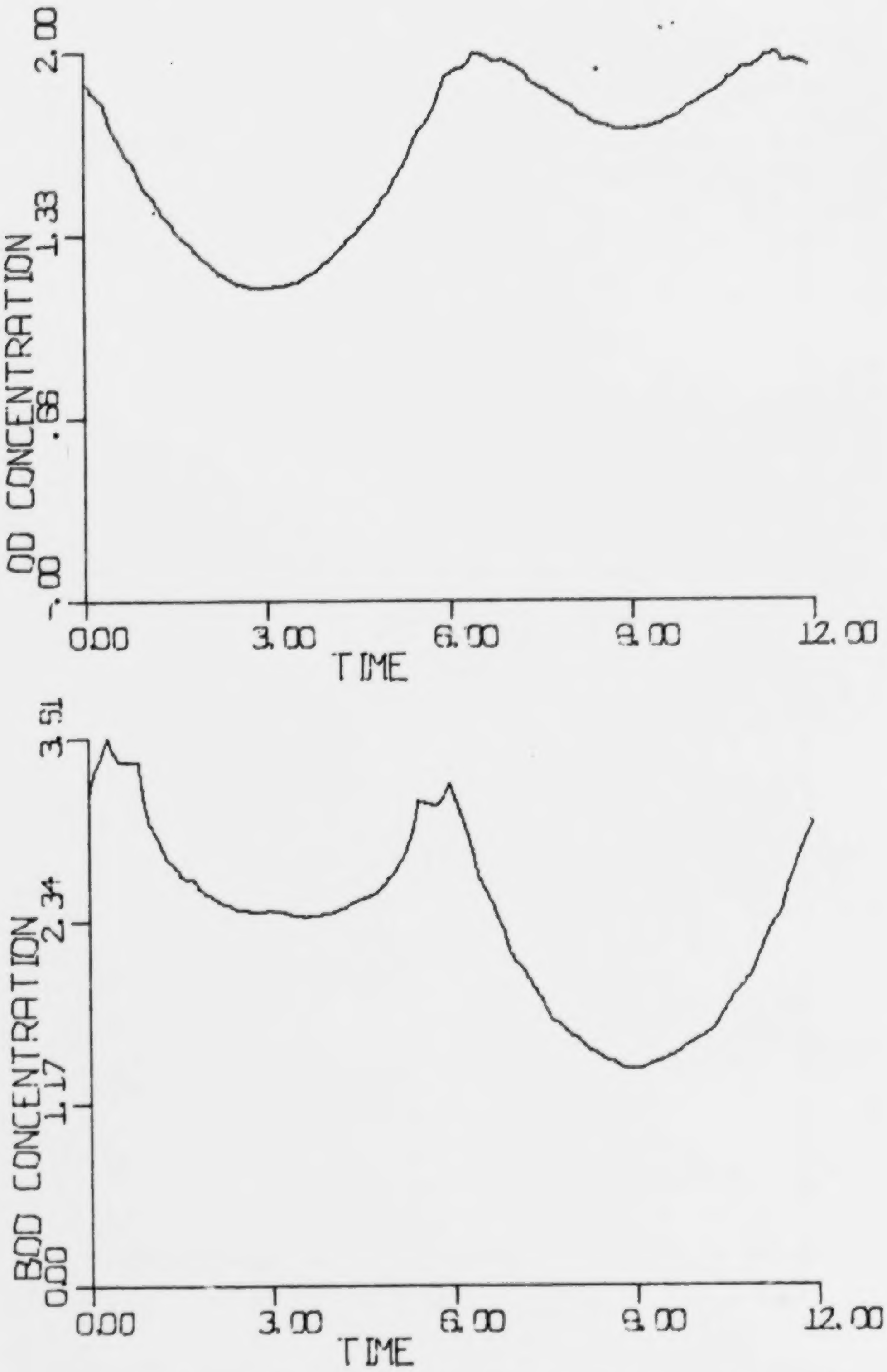
NON-DISPERSION MODEL . LINE SOURCE, LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE FLOOD TIDE

FIGURE 21



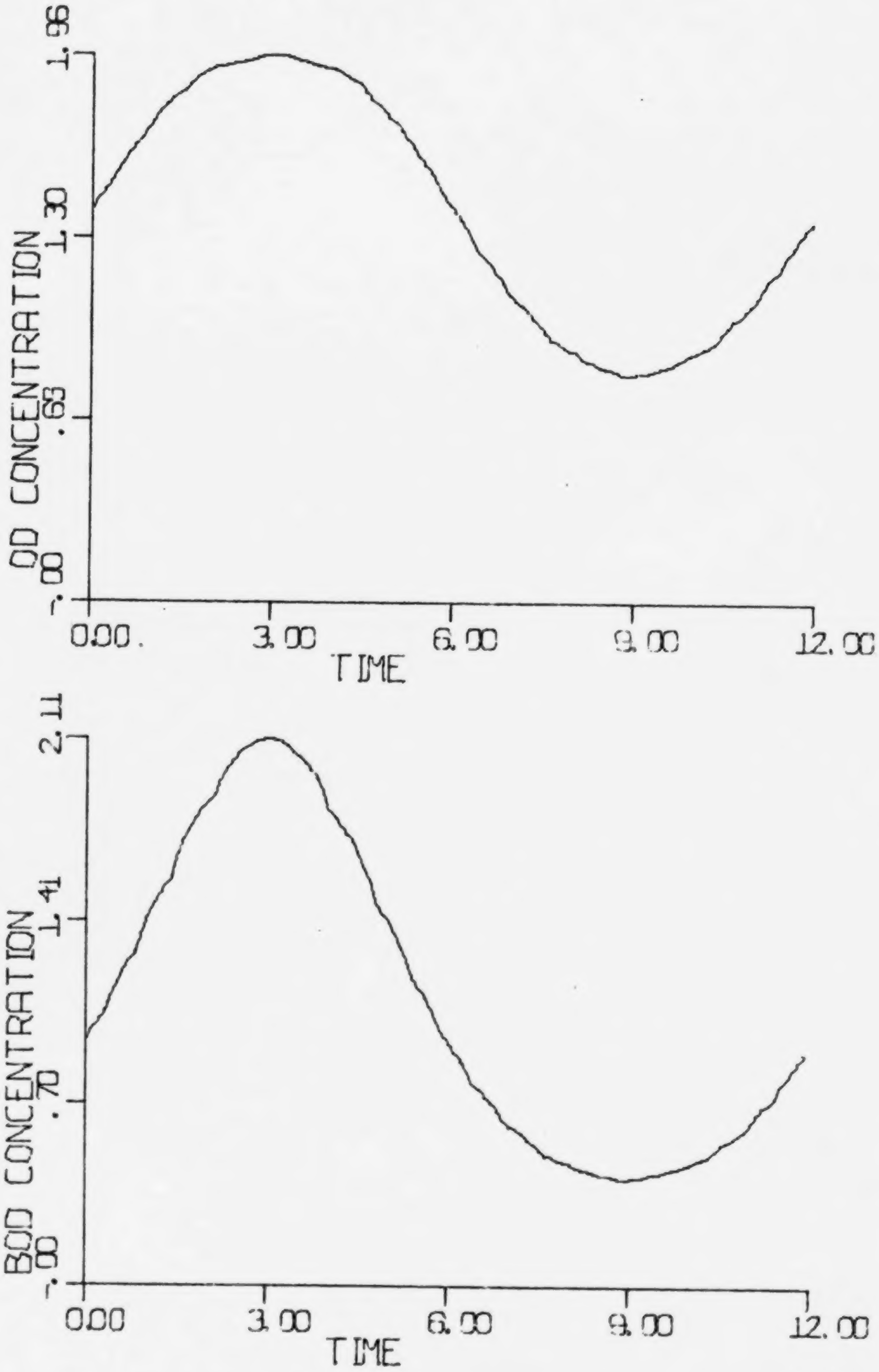
NON-DISPERSION MODEL - LINE SOURCE, LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 22



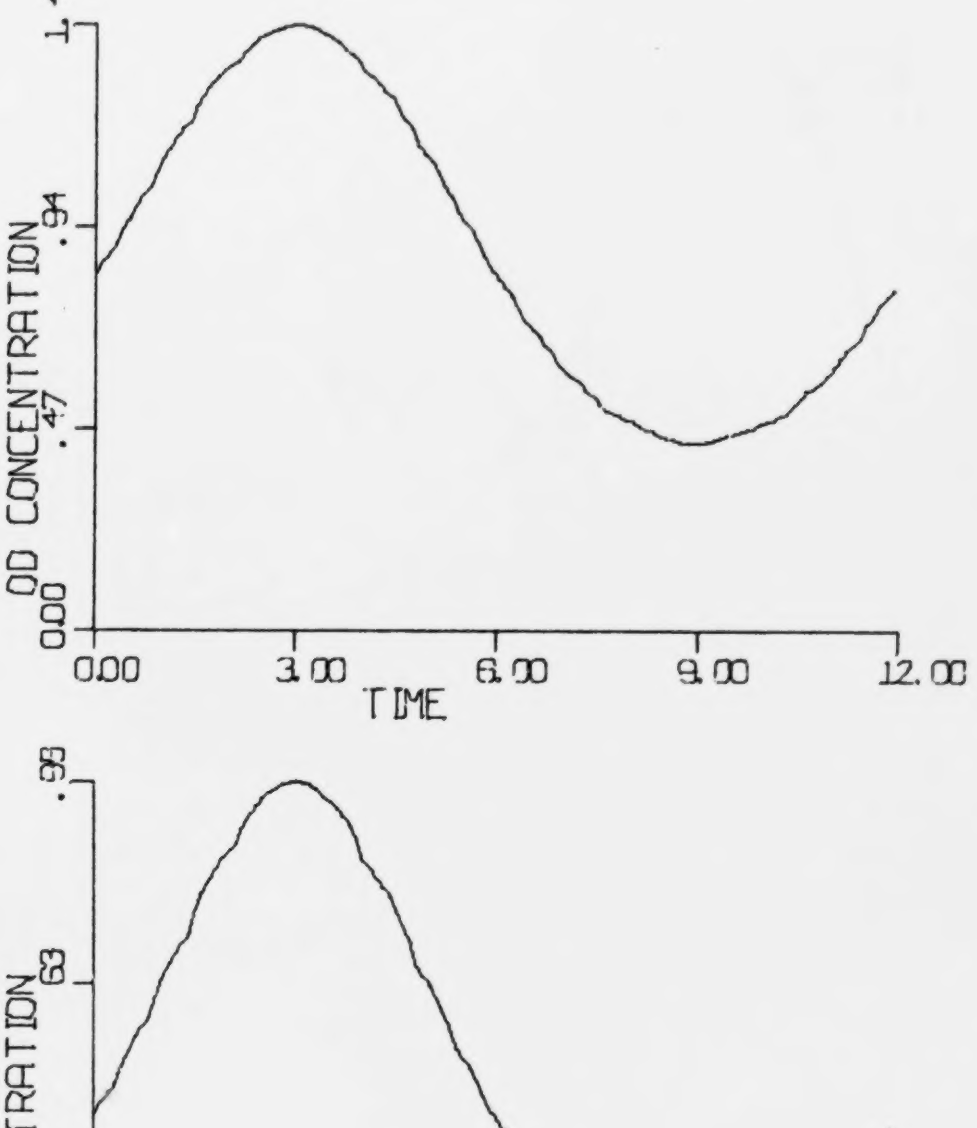
NON-DISPERSION MODEL - LINE SOURCE. LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM

FIGURE 23



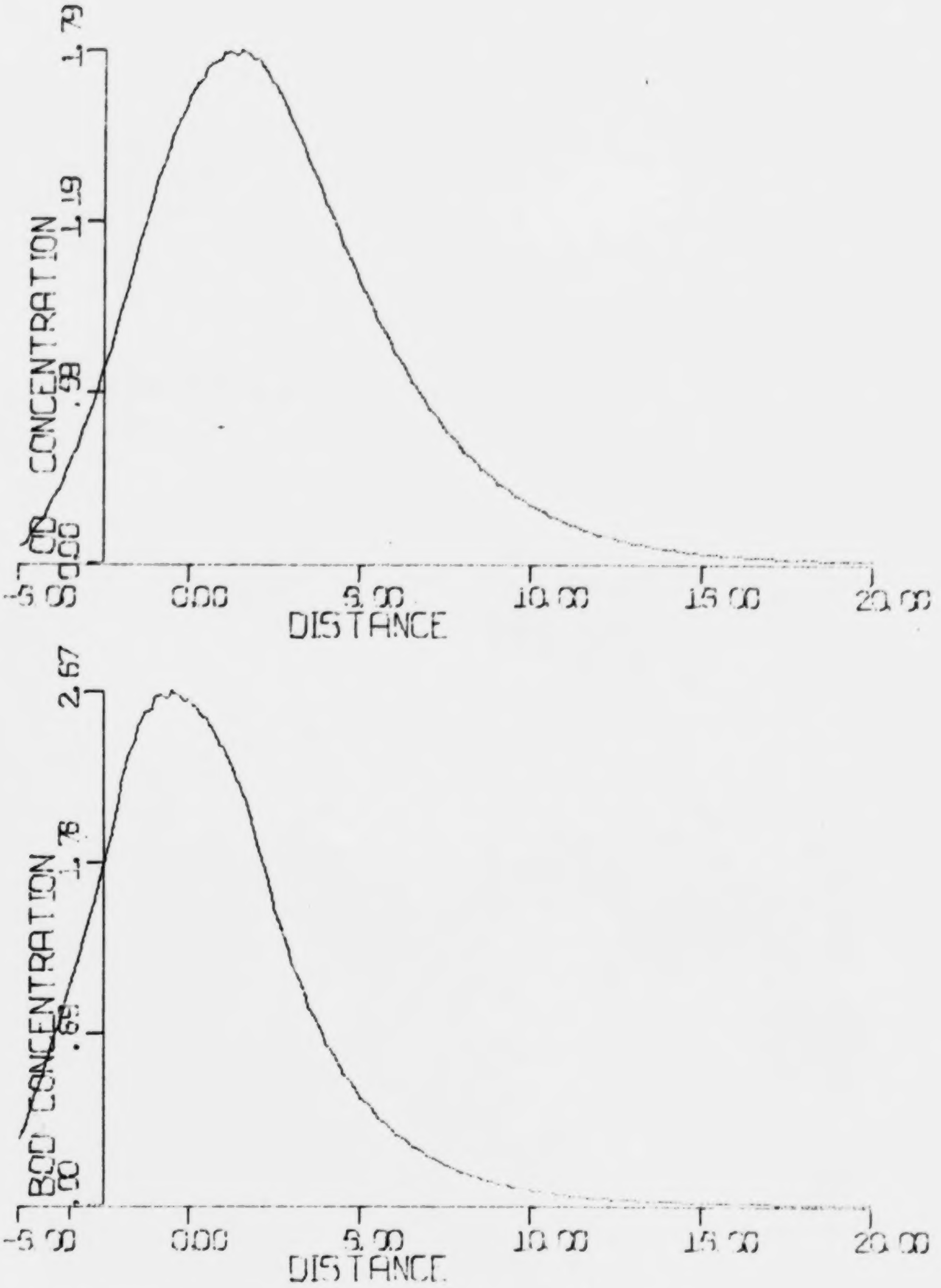
NON-DISPERSION MODEL - LINE SOURCE, LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 24



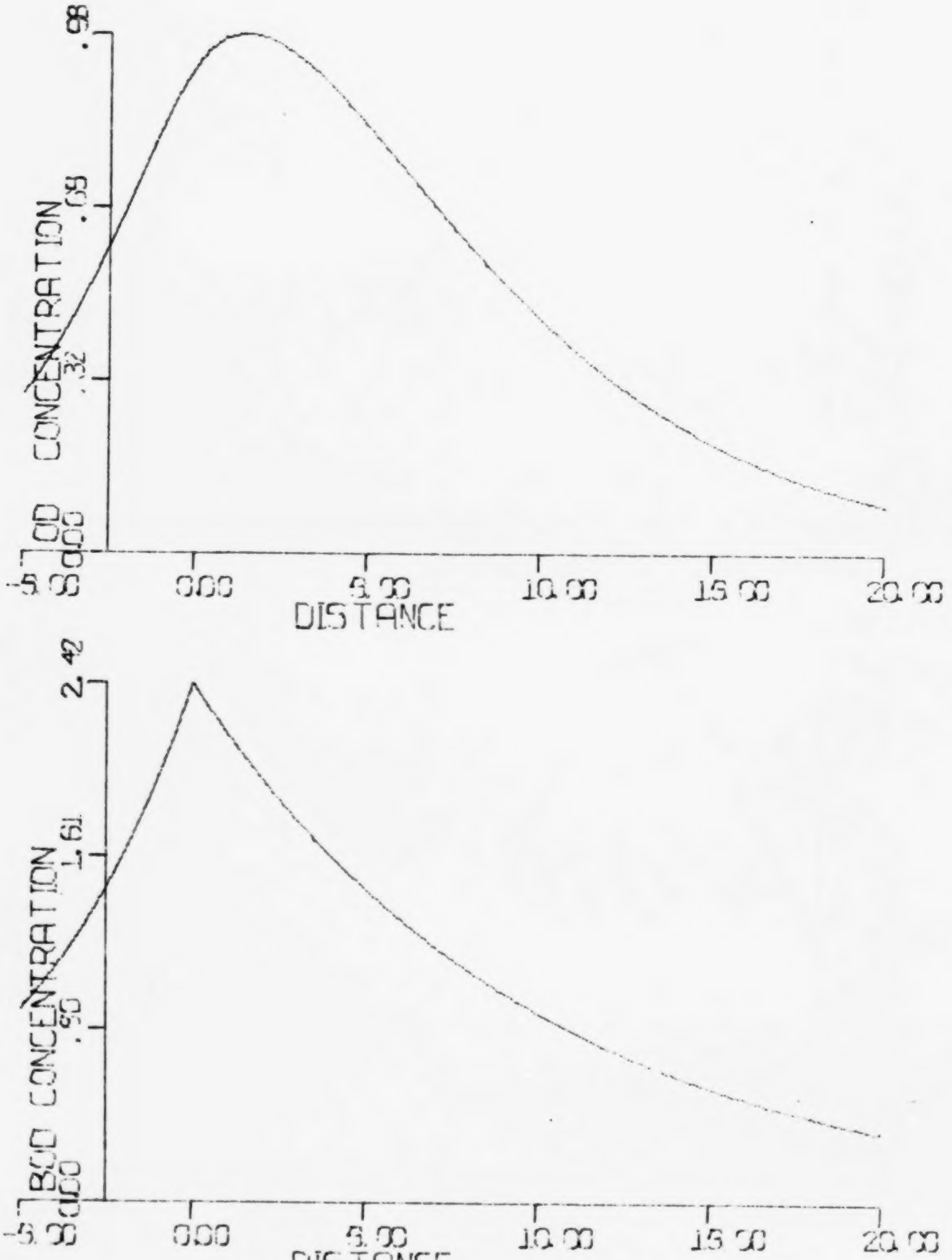
NON-DISPERSION MODEL - LINE SOURCE. LENGTH ONE MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 25



NON-DISPERSION MODEL - LINE SOURCE, LENGTH FOUR MILES
BOD. AND DO. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE

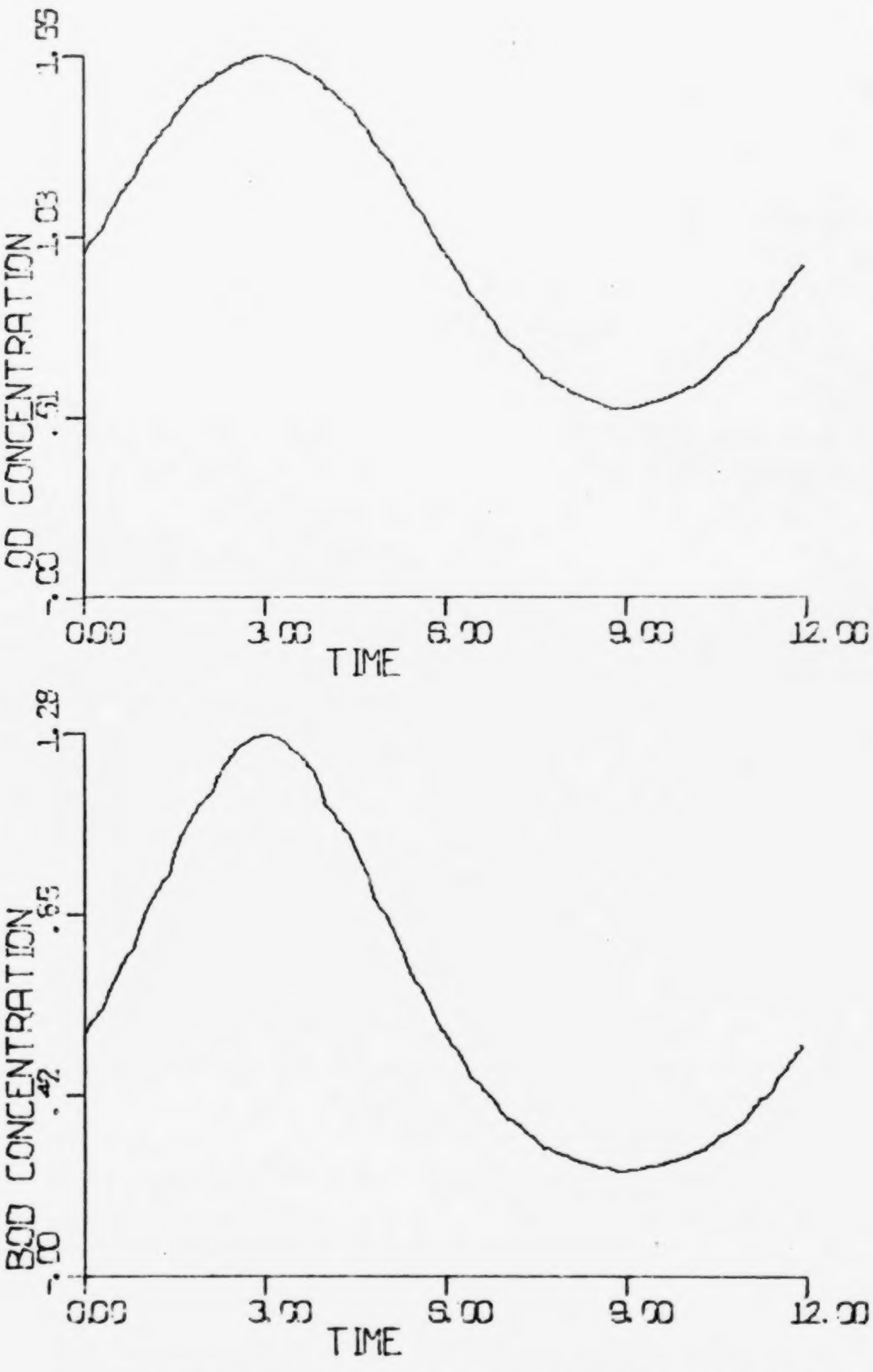
FIGURE 25



OCONNOR'S MODEL

DIFFUSION COEFFICIENT - FIVE SQMI PER DAY

FIGURE 27



NON-DISPERSION MODEL - LINE SOURCE, LENGTH FOUR MILES
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 28

DIFFUSION MODEL

Introduction

In the preceding chapter stochastic predictions for estuary conditions were obtained. The effect of dispersion was neglected, leading to a relatively elementary mathematical analysis. The introduction of a dispersion component leads to a more formidable analysis, but the difficulties are not insurmountable. A model is developed in this chapter which considers diffusion as an additional random process. The diffusion phenomenon in this model represents the true diffusion effect; not the effect confounded with tidal motion. For this reason the term diffusion is used in lieu of dispersion. As in the earlier model, the time variable velocity is treated as a separate mechanism.

The physical limitations placed upon the system are similar to those for the Non-Dispersion Model, which are outlined in Scope and Limitations. The stochastic nature of the estuary mechanisms are as assumed by Thayer and Krutchkoff (59, 60).

The model considers all factors considered by Dobbins in his deterministic work, and consequently by Thayer and Krutchkoff, except

a) L_a the uniform increase of BOD by land runoff,

b) D_B the decrease in DO due to the benthal demand.

These factors can be handled separately, and their effect added to the results of this model by the principle of superposition. This is done in Land Run Off and Benthal Demand.

As in the work of Thayer and Krutchkoff, the pollution and dissolved oxygen are discretized into units of size Δ. A Δ unit is such that it cannot be further decomposed into smaller units. Pollution must be degraded by the bacteria and other mechanisms in units of size Δ or multiples of Δ. Likewise, DO is decreased or increased by units of size Δ or multiples of Δ.

First, the motion of a single BOD unit is considered. This motion will be treated as a random walk with decay on the real axis. In each time interval δt, the unit has probability $p(t)$ of taking a positive step of length δx, probability $q(t)$ of taking a negative step of length δx, and probability r of being absorbed. A difference equation is obtained which gives the joint probability of the unit being at any location x after any time t, given that the unit was at the origin at time t_0 (if x is not a multiple of δx this probability is zero). $p(t)$, $q(t)$, r, δx, and δt can be determined from the physical parameters of the estuaries.

The diffusing process of a unit of oxygen deficit is handled in a similar manner, with the initial conditions slightly modified. Unlike the BOD units which originate at a uniform rate, a unit of OD is formed simultaneously with the absorption of a unit of BOD.

The parameter δt used in the random walk procedure is the length of time needed for one unit of size Δ to enter the estuary. Thus, δt is a function of the rate at which pollution enters from the source, and the cross-sectional area of the estuary. In order to obtain the effect of a continuous source, units of size Δ are released at each time interval δt. The units act independently and their behavior can be studied individually by the random walk mechanism. Thus, the probability of a given concentration at location x after time t is the convolution of the probabilities associated with the individual units.

The resulting model is a transient state model; the time parameter being defined as the time lapse since the source of pollution originated.

Due to the fact that the velocity of the estuary is a function of time, the distribution will be continually changing in time and no steady state is possible. However, after a sufficient time lapse the distribution for a given point in the tidal cycle remains constant from cycle to cycle. This will be defined as a pseudo-steady state condition. The pseudo-steady state can be described by taking the time lapse to be sufficiently large. In most cases 15 to 25 days will be 99% of pseudo-steady state.

The probability distribution is a function of the location in the estuary and the time in the tidal cycle; thus a great many distributions are necessary to adequately describe the situation. For example, if one would like to consider a 50-mile stretch of the estuary over an entire tidal cycle, a reasonable procedure would be to consider each 10th of a mile at time intervals of .1 hours. This would require 60,000 separate distributions. This introduces the restraint that each distribution must be obtained in a minimum of time (computer time). The discrete random walk approach gives results which can be obtained directly from a computer. The running times for the programs are a function of the number of steps of size δt which are needed to reach the time of interest. In general δt is very small and the time required to obtain one distribution approaching steady state by this procedure would be prohibitive. If one wished to obtain many such distributions the above model would have to be discarded as impractical.

Fortunately, the fact that δt is small can be used to advantage. At each stage continuous results will be obtained by letting δt approach zero. Since δt is small, these results represent a very good approximation to the discrete case.

Difference Equations

The motion of a single particle placed in a medium where it is exposed to a great number of random forces is described by what is commonly called "diffusion." The diffusion process considered in this analysis was first developed by Einstein in relation to the Brownian motion problem. Literature concerning the Brownian motion problem is extensive. Heuristic developments of the limiting process leading to the renowned Fokker-Planck Equation are given by Feller (14), Rosenblatt (53), Bharucha-Reid (7) and others (6,33,50). More advanced treatments are rendered by Feller (16,17,18,19), Doob (12) and Kac (32). Geneticists have been particularly active in investigating the stochastic nature of the diffusion phenomenon (15,37). Although this is not the only mathematical description of the diffusion process it is the most widely used and verified. The general treatment concerns the motion of a conservative substance. Here the process is modified for a non-conservative process.

The general assumption for this process is that it is certain that the particle will move in any interval of time δt; however, if δt is small the movement will be small. The forces acting on the particle can be summarized as two components: the symmetric random forces described as diffusion, and the non-symmetric drift, or in this case, velocity. In the estuary velocity is a function of time. In addition, when working with non-conservative substances, such as BOD or DO, the possibility of absorption must be included.

The motion of a BOD unit is characterized as follows: in each time lapse of length δt the unit moves forward on the positive axis a distance δx with probability $p(t)$, or backwards the distance δx with probability $q(t)$. This is the only motion possible. However, at each time lapse of length δt the unit may be absorbed and lost to the system. The probability of absorption is constant r. $p(t) + q(t) + r = 1$ for all t.

Let u(m,n) be the probability that after n steps the unit of BOD is at location $m\delta x$. The total time lapse t will be $n\delta t$.

The unit may be at location $m\delta x$ after n steps in either of two ways:

1. the unit is at $(m-1)\ \delta x$ at $(n-1)\delta t$ and takes a positive step, or
2. the unit is at $(m+1)\delta x$ at $(n-1)\delta t$ and takes a negative step.

Thus

$$u(m,n+1) = u(m-1,n)\, p(n\,\delta t) + u(m+1,n)\, q(n\,\delta t) \tag{4.1}$$

for

$$m,n \in I,\ -\infty < m < \infty\ , \text{and } 0 \leqslant n < \infty$$

Remaining consistent with the physical interpretation of the Brownian motion process, the following values for the terms in Equation (4.1) are adopted:

$$\frac{\delta x^2}{\delta t} = 2E\ , \tag{4.2}$$

$$p(t) = (1 - K_d\ \delta t)/2 + \frac{U(t)}{2}\ \frac{\delta t}{\delta x}\ , \tag{4.3}$$

$$q(t) = (1 - K_d\ \delta t)/2 - \frac{U(t)}{2}\ \frac{\delta t}{\delta x}\ , \tag{4.4}$$

$$r = K_d\ \delta t \tag{4.5}$$

where

E is the diffusion coefficient,

K_1 is the coefficient of deoxygenation,

K_3 is the coefficient corresponding to all non-oxygen consuming processes,

K_d $= K_1 + K_3$,

and

$U(t)$ is the velocity function.

δt is defined as the length of time required for one unit of size Δ to enter the estuary, and may be calculated from physical parameters. With this value of δt and the relations given above, all components in the difference equation may be evaluated.

In order to solve equation (4.1), a set of initial conditions is required. Given that the unit entered the system at time t_o (t_o must be divisible by δt), the initial conditions are

$$u(0,n_o) = 1 \ ,$$

$$u(m,n_o) = 0 \text{ for } m \neq 0 \ ,$$

$$u(m,n) = 0 \text{ for } n < n_o \ ,$$

where

$$n_o = t_o / \delta t \ . \tag{4.6}$$

Under these initial conditions $u(m,n)$ will be written $u(m,n | n_o)$.

To derive the difference equation for oxygen deficit, it must be recalled that the absorption of a unit of BOD generates a unit of OD of equal size. The OD is then absorbed from the system by reaeration in the same manner that BOD is absorbed by the bacteria.

One can think of a unit being released at t_o as BOD. The BOD travels in accordance with the random walk mechanism until it is absorbed; however, the absorption is only a metamorphosis which changes the identity of the BOD to OD. The renamed unit then resumes the random walk at the point where it was absorbed, now governed by the probabilities associated with OD.

Let $v(m,n)$ be the probability that a unit of OD is at location $m\delta x$ after time $n\delta t$, where δx and δt are as before. The probability of a forward step will be represented by $\underline{p}(t)$, a negative step by $\underline{q}(t)$, and absorption by $\underline{r}$ An OD unit can be at $m\delta x$ after time $n\delta t$ in any of three ways:

1. the unit is at $(m-1)\delta x$ at $(n-1)\delta t$ and takes a step forward,

2. the unit is at $(m+1)\delta x$ at $(n-1)\delta t$ and takes a step backwards,

3. the unit is BOD located at $m\delta x$ at $(n-1)\delta t$, and is transformed into OD.

Thus,

$$v(m,n+1) = v(m-1,n)\ \underline{p}\ (n\delta t) + v(m+1,n)\ \underline{q}\ (n\delta t) + u(n,m)\ K_1\ \delta t \tag{4.7}$$

for

$$m,n \in I,\ -\infty < m < \infty, \text{ and } 0 \leqslant n < \infty .$$

The transition probabilities are defined as follows:

$$\underline{p}\,(t) = (1\text{-}K_2\ \delta t)/2 + \frac{U\,(t)}{2}\ \frac{\delta t}{\delta x}\ , \tag{4.8}$$

$$\underset{\sim}{q}(t) = (1-K_2\,\delta t)/2 - \frac{U(t)}{2}\,\frac{\delta t}{\delta x}\,, \tag{4.9}$$

$$\underset{\sim}{r} = K_2\,\delta t\,, \tag{4.10}$$

where K_2 is the coefficient of reaeration, and the other parameters are defined following Equation (4.5).

The initial condition is

$$v(m,n) = 0 \text{ for all } m \text{ if } n < n_o\,. \tag{4.11}$$

$v(m,n \mid n_o)$ will represent $v(m,n)$ under condition (4.11).

Numerical solutions to Equations (4.1) and (4.7) can be obtained by either of two methods. The solution for any m and n can be obtained iteratively. That is, starting at n_o the solution can be found for all m and $n_o + 1$. Using these solutions the set of solutions for $n_o + 2$ can be obtained. This process can be continued until any desired m and n are reached.

The second method utilizes generating functions. If

$$U(s,n \mid n_o) = \sum_{m=-\infty}^{\infty} u(m,n \mid n_o)\, s^m\,, \tag{4.12}$$

and

$$V(s,n \mid n_o) = \sum_{m=-\infty}^{\infty} v(m,n \mid n_o)\, s^m, \tag{4.13}$$

then

$$U(s,n|n_o) = s^{-(n-n_o)} \prod_{i=n_o}^{n-1} [p(i\delta t)s^2 + q(i\delta t)] , \qquad (4.14)$$

and

$$V(s,n|n_o) = r \sum_{i=n_o}^{n-1} U(s,i|n_o)s^{-(n-i+1)} \prod_{j=i+1}^{n-1} [\underline{p}(j\delta t)s^2 + \underline{q}(j\delta t)] . \qquad (4.15)$$

The desired values of $u(m,n|n_o)$ and $v(m,n|n_o)$ are found by picking out the coefficient of s^m in $U(s,n|n_o)$ and $V(s,n|n_o)$, respectively.

Fortran IV programs for both of these methods appear in Appendix B. However, for small δt, both programs require large amounts of time. In order to obtain solutions near steady state up to 8,000 steps are required. The first method calculates all of the solutions for each value of n up to the value desired. The number of values of m for which u(m,n) and v(m,n) are non-zero increases as n increases. Thus, running time increases rapidly as n increases.

The program which utilizes generating functions calculates $p(i\delta t)$, $q(i\delta t)$, $p(i\delta t)$ and $q(i\delta t)$ for $i = n_o$ to n. A subprogram is called which multiplies binomials. For BOD the number of binomials is $(n-n_o)$. For OD, $(n-n_o)$ binomials must be multiplied for each term in the summation of $V(s,n|n_o)$, i.e., binomial multiplication must be called $(n-n_o)$ times. Again the running time of this program increases rapidly as n increases.

Solutions of difference equations (4.1) and (4.7) were obtained for small values of n. No attempt was made to use these programs for values of n approaching steady state.

Continuous Approximations to Difference Equations

In the preceding work practical difficulties arose due to the small size of δt. Fortunately this can be used to advantage. As δt approaches zero, limiting results which approximate $u(m,n|\, n_o)$ and $v(m,n|\, n_o)$ can be obtained.

As it approaches zero the probabilities $u(m,n|\, n_o)$, and $v(m,n|\, n_o)$ approach continuous densities. These densities will be denoted by $f(x,t|\, t_o)$ and $g(x,t|\, t_o)$, respectively. Recalling that the location on the real axis corresponding to m is $m\delta x$, and the time corresponding to the n^{th} step is $n\delta t$, Equations (4.1) and (4.7) can be rewritten as follows:

$$f(x,t + \delta t|\, t_o) = f(x-\delta x, t|\, t_o)\, p(t) + f(x+\delta x, t|\, t_o)\, q(t), \tag{4.16}$$

$$g(x,t + \delta t|\, t_o) = g(x-\delta x, t|\, t_o)\, \underline{p}(t) + g(x+\delta x, t|\, t_o)\, \underline{q}(t)$$

$$+ f(x,t|\, t_o)\, K_1\, \delta t\ . \tag{4.17}$$

Expanding the terms in (4.16) in a Taylor series one obtains

$$f(x,t|\, t_o) + \delta t \frac{\partial f}{\partial t} + \frac{(\delta t)^2}{2} \frac{\partial^2 f}{\partial t^2} + o(\delta t^3)$$

$$= p(t)\ [f(x,t|\, t_o) - \delta x \frac{\partial f}{\partial x} + \frac{\delta x^2}{2} \frac{\partial^2 f}{\partial x^2} + o(\delta x^3)\ ]$$

$$+ q(t)\ [\ f(x,t|\, t_o) + \delta x \frac{\partial f}{\partial x} + \frac{\delta x^2}{2} \frac{\partial^2 f}{\partial x^2} + o(\delta x^3)\ ]\ . \tag{4.18}$$

Regrouping terms, the right side of Equation (4.18) becomes

$$f(x,t|t_o)\ [q(t) + p(t)] + \delta x \frac{\partial f}{\partial x} [q(t) - p(t)]$$

$$+ \frac{\delta x^2}{2} \frac{\partial^2 f}{\partial x^2} [p(t) + q(t)] + o(\delta x^3) . \qquad (4.19)$$

Recall that

$$2E = \frac{\delta x^2}{\delta t} ,$$

$$p(t) + q(t) = 1 - K_d\ \delta t ,$$

and

$$p(t) - q(t) = U(t) \frac{\delta t}{\delta x} .$$

Therefore,

$$\frac{\partial f}{\partial t} + \frac{\delta t}{2} \frac{\partial^2 f}{\partial t^2} + o(\delta t^2)$$

$$= E \frac{\partial^2 f}{\partial x^2} - U(t) \frac{\partial f}{\partial x} - K_d f(x,t|t_o) + o(\delta x^2) . \qquad (4.20)$$

Therefore as $\delta t \rightarrow 0$ and $\delta x \rightarrow 0$, Equation (4.20) becomes

$$\frac{\partial f(x,t|t_o)}{\partial t} = E \frac{\partial^2 f}{\partial x^2} - U(t)\frac{\partial f}{\partial x} - K_d f(x,t|t_o). \qquad (4.21)$$

Similarly, Equation (4.17) becomes

$$\frac{\partial g(x,t|t_o)}{\partial t} = E \frac{\partial^2 g}{\partial x^2} - U(t) \frac{\partial g}{\partial x} - K_2 g(x,t|t_o) + K_1 f(x,t|t_o) .$$

$$(4.22)$$

Equations (4.21) and (4.22) are identical to those obtained by O'Connor, with the constant fresh water velocity term replaced by a time variable term. However, there are fundamental differences in the interpretation of the variables. In the equations of O'Connor, the solutions to Equations (4.21) and (4.22) represent the BOD and OD concentrations. In this analysis f and g are the probability densities corresponding to a single unit of BOD and OD being located at x at time t. In order to obtain statements concerning the concentrations at any point in time and space it is necessary to resort to a convoluting procedure. This procedure will be described in the next section.

In order to solve these partial differential equations it will be necessary to prove the following lemmas:

Lemma 1 : Solutions to the set of partial differential equations

$$\frac{\partial f(x,t)}{\partial t} = E \frac{\partial^2 f(x,t)}{\partial x^2} - U(t) \frac{\partial f(x,t)}{\partial x} - K_d f(x,t) , \qquad (4.23)$$

and

$$\frac{\partial g(x,t)}{\partial t} = E \frac{\partial^2 g(x,t)}{\partial x^2} - U(t) \frac{\partial g(x,t)}{\partial x} - K_2 g(x,t) + K_1 f(x,t) , \quad (4.24)$$

are given by

$$f(x,t) = f(v(t), t) , \quad (4.25)$$

and

$$g(x,t) = g(v(t), t) , \quad (4.26)$$

where *f* and *g* are defined as solutions to

$$\frac{\partial f(x,t)}{\partial t} = E \frac{\partial^2 f(x,t)}{\partial x^2} - K_d f(x,t) , \quad (4.27)$$

and

$$\frac{\partial g(x,t)}{\partial t} = E \frac{\partial^2 g(x,t)}{\partial x^2} - K_2 g(x,t) + K_1 f(x,t), \quad (4.28)$$

and

$$v(t) = x - \int_{t_o}^{t} u(\tau)\, d\tau \ , \ t_o \text{ constant.} \tag{4.29}$$

Proof: Let $f(x,t) = \mathit{f}(v(t),t)$, then

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial \mathit{f}(v,t)}{\partial t} = \frac{\partial \mathit{f}}{\partial v}\,\frac{\partial v}{\partial t} + \frac{\partial \mathit{f}}{\partial t} . \tag{4.30}$$

But

$$\frac{\partial v}{\partial t} = - U(t) \ . \tag{4.31}$$

Therefore,

$$\frac{\partial f(x,t)}{\partial t} = - U(t)\,\frac{\partial \mathit{f}}{\partial v} + \frac{\partial \mathit{f}}{\partial t} . \tag{4.32}$$

Also,

$$\frac{\partial f(x,t)}{\partial x} = \frac{\partial \mathit{f}}{\partial x} = \frac{\partial \mathit{f}}{\partial v}\,\frac{\partial v}{\partial x} = \frac{\partial \mathit{f}}{\partial v} \ , \tag{4.33}$$

since

$$\frac{\partial v}{\partial x} = 1 \ . \tag{4.34}$$

Similarly,

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial^2 f}{\partial v^2} \ . \tag{4.35}$$

Substituting Equations (4.32) through (4.35) into (4.23), yields

$$- U(t) \frac{\partial f(v,t)}{\partial v} + \frac{\partial f(v,t)}{\partial t} = E \frac{\partial^2 f(v,t)}{\partial v^2} - U(t) \frac{\partial f(v,t)}{\partial v}$$

$$- K_d f(v,t) \ , \tag{4.36}$$

i.e.,

$$\frac{\partial f(v,t)}{\partial t} = E \frac{\partial^2 f(v,t)}{\partial v^2} - K_d f(v,t) \ . \tag{4.37}$$

which is identical to (4.27).

Similarly, let $g(x,t) = g(v,t)$. Substitute the derivatives of $g(x,t)$ in terms of $g(v,t)$ into Equation (4.24), the resulting equation will be (4.28). q.e.d.

In order to solve Equations (4.21) and (4.22) one need only solve the simpler set of equations

$$\frac{\partial f(x,t|t_o)}{\partial t} = E \frac{\partial^2 f(x,t|t_o)}{\partial x^2} - K_d f(x,t|t_o) , \tag{4.38}$$

and

$$\frac{\partial g(x,t|t_o)}{\partial t} = E \frac{\partial^2 g(x,t|t_o)}{\partial x^2} - K_2 g(x,t|t_o) + K_1 f(x,t|t_o) , \tag{4.39}$$

and translate the results on the x-axis by a distance

$$- \int_{t_o}^{t} U(\tau)\, d\tau .$$

Lemma 2 :

$$\int_{-\infty}^{\infty} f(x,t|t_o)\, dx = e^{-K_d (t-t_o)} \text{ for all } t > 0 , \tag{4.40}$$

where $f(x,t)$ is the density function for BOD as given above.

Proof: $\int_{-\infty}^{\infty} f(x,t|\,t_o)\,dx$ = Pr [the BOD unit is still in existence at time t, given that it entered the system at t_o]. Recall that the probability that the BOD is absorbed in a time interval δt is $K_d \delta t$.

Let p_n = Pr[the unit of BOD is still in existence after n time units of length δt, given that it entered the system after n_o steps]. Therefore

$$p_n = p_{n-1}\,(1-K_d\;\delta t),$$

and

$$p_{n_o} = 1, \text{ where } n_o = t_o/\Delta t\,. \tag{4.41}$$

Thus,

$$p_n = (1-K_d\,\delta t)^{\,n-n_o}\;. \tag{4.42}$$

Now,

$$t = n\,\delta\,t\;\;, \tag{4.43}$$

and

$$p_n = (1 - K_d\,\delta t)^{(t-t_o)/\delta t} \qquad (4.44)$$

From this, one obtains

$$\lim_{n \to \infty} p_n = \int_{-\infty}^{\infty} f(x,t \mid t_o) / dx \;, \qquad (4.45)$$

and

$$\lim_{n \to \infty} p_n = \lim_{\delta t \to 0} (1 - K_d\,\delta t)^{(t-t_o)/\delta t} = e^{-K_d (t-t_o)} . \qquad (4.46)$$

q.e.d.

The initial conditions necessary for the solution of Equation (4.38) are

1. $$\int_{-\infty}^{\infty} f(x,t \mid t_o)\,dx = e^{-K_d (t-t_o)} \text{ for all } t > t_o . \qquad (4.47)$$

2. $$\lim_{t \to t_o^+} f(x,t \mid t_o) = 0 . \qquad (4.48)$$

3. $$\lim_{x \to 0^-} f(x,t \mid t_o) = \lim_{x \to 0^+} f(x,t \mid t_o) \ . \qquad (4.49)$$

4. $$\lim_{x \to \infty} f(x,t \mid t_o) = \lim_{x \to -\infty} f(x,t \mid t_o) = 0 \ . \qquad (4.50)$$

The first initial condition is Lemma 2. The second condition is a result of the initial condition on the difference equation. The third condition requires that $f(x,t|t_o)$ be continuous and the fourth condition arises from the fact that the integral in condition one is bounded.

The variable transformation $t \to t - t_o$ can be made. This will simplify the mechanics of the solution. Under this transformation initial conditions (4.47), (4.48) and (4.49) become

1. $$\int_{-\infty}^{\infty} f(x,t)\,dx = e^{-K_d t} \ , \ t > 0 \qquad (4.51)$$

and

2. $$\lim_{t \to 0^-} f(x,t) = \lim_{t \to 0^+} f(x,t) = 0 \ , \text{ where } f(x,t \mid 0) = f(x,t) \ . \qquad (4.52)$$

Equation (4.38) can now be solved by the use of Laplace transforms. Define $\pi(x,s)$ as the Laplace transform of $f(x,t)$ with respect to t, i.e.,

$$\pi(x,s) = \int_0^\infty f(x,t)\ e^{-st}\, dt\ .$$

Restating Equation (4.38), one has

$$\frac{\partial f(x,t)}{\partial t} = E\ \frac{\partial^2 f(x,t)}{\partial x^2} - K_d f(x,t)\ .$$

Taking the Laplace transform of Equation (4.38) one obtains the ordinary differential equation

$$s\,\pi(x,s) - f(x,0^+) = E\ \frac{\partial^2 \pi(x,s)}{\partial x^2} - K_d \pi(x,s)\ . \qquad (4.53)$$

By initial condition (4.52) $f(x,0^+) = 0$, and therefore

$$s\pi(x,s) = E\ \frac{\partial^2 \pi(x,s)}{\partial x^2} - K_d \pi(x,s)\ . \qquad (4.54)$$

Equation (4.54) can be solved by conventional methods as follows:

Suppose

$$\pi(x,s) = e^{mx}\ , \qquad (4.55)$$

then

$$Em^2 - (s+K_d) = 0\ . \qquad (4.56)$$

From this one obtains

$$m_1 = \sqrt{\frac{s+K_d}{E}} \; ; \qquad m_2 = -\sqrt{\frac{s+K_d}{E}} \qquad (4.57)$$

as solutions for m. Thus, the general solution to Equation (4.54) is

$$\pi(x,s) = c_1 e^{m_1 x} + c_2 e^{m_2 x} \, , \qquad (4.58)$$

where c_1 and c_2 are constants. However, m_1 is positive and m_2 is negative, thus by initial condition (4.52),

$$c_1 = 0 \text{ if } x > 0 \, ,$$

and

$$c_2 = 0 \text{ if } x < 0 \, . \qquad (4.59)$$

This implies that

$$\pi(x,s) = \begin{cases} c_1 e^{m_1 x} & \text{when } x < 0 \, , \\ c_2 e^{m_2 x} & \text{when } x > 0 \, . \end{cases} \qquad (4.60)$$

Also initial condition (4.52) implies that $c_1 = c_2 = c$, yielding

$$\pi(x,s) = \begin{cases} c\, e^{m_1 x} \text{ when } x \leqslant 0 \\ \\ c\, e^{m_2 x} \text{ when } x \geqslant 0 \end{cases} \tag{4.61}$$

By taking the Laplace transform of both sides of initial condition (4.51), one obtains

$$\int_{-\infty}^{\infty} \pi(x,s)\, dx = \frac{1}{K_d + s} \tag{4.62}$$

But from Equation (4.61) one has

$$\int_{-\infty}^{\infty} \pi(x,s)\, dx = c \int_{-\infty}^{0} e^{m_1 x}\, dx + c \int_{0}^{\infty} e^{m_2 x}\, dx = 2c \sqrt{\frac{E}{s + K_d}}. \tag{4.63}$$

Solving for c yields

$$c = (4E(s + K_d))^{-\frac{1}{2}} \quad . \qquad (4.64)$$

Thus,

$$\pi(x,s) = \begin{cases} (4E\,(s+K_d))^{-\frac{1}{2}}\; e^{+x\sqrt{\frac{s+K_d}{E}}} & \text{if } x \leqslant 0 \,, \\ (4E\,(s+K_d))^{-\frac{1}{2}}\; e^{-x\sqrt{\frac{s+K_d}{E}}} & \text{if } x > 0 \,. \end{cases} \qquad (4.65)$$

This can be inverted with the aid of a table of Laplace transforms to give

$$f(x,t) = \frac{1}{2\sqrt{\pi E t}}\; e^{\frac{-x^2}{4Et} - K_d t} \qquad (4.66)$$

Taking the inverse transformation $t \rightarrow t - t_o$, the solution to Equation (4.38) is

$$f(x,t \mid t_o) = \frac{1}{2\sqrt{\pi E(t-t_o)}} e^{\frac{-x^2}{4E(t-t_o)} - K_d(t-t_o)} \quad . \quad (4.67)$$

Therefore by Lemma 1, the solution to Equation (4.21) is

$$f(x,t \mid t_o) = \frac{1}{2\sqrt{\pi(t-t_o)}} e^{\frac{-(x-v(t))^2}{4E(t-t_o)} - K_d(t-t_o)} \quad , \quad (4.68)$$

where

$$v(t) = \int_{t_o}^{t} U(\tau)\, d\tau \; .$$

The solution to Equation (4.39) and consequently to Equation (4.22) is approached in the following manner.

Lemma 3 :

$$\int_{-\infty}^{\infty} g(x,t \mid t_o)\, dx = \frac{K_1}{K_2 - K_d} \left[e^{-K_d(t-t_o)} - e^{-K_2(t=t_o)} \right] \quad (4.69)$$

for all $t > t_o$, where $g(x,t|t_o)$ is defined in Equation (4.22),

Proof : $\int_{-\infty}^{\infty} g(x,t|t_o)\,dx =$ Pr[A DO deficit unit is in existence at time t, given that its ancestral BOD unit entered the system at t_o].

Let q_n = Pr[The unit of OD is in existence after n time intervals of length δt, given that its ancestral BOD entered the system after n_o steps]. The difference equation for q_n is

$$q_{n+1} = q_n\,(1-K_2\,\delta t) + p_n\,K_1\,\delta t\ , \tag{4.70}$$

where p_n is defined in Lemma 2.

The solution to this difference equation is

$$q_n = \sum_{i=n_o}^{n-1} (1-K_d\,\delta t)^{i-n_o}\,(1-K_2\,\delta t)^{n-i-1}\,K_1\,\delta t$$

$$= K_1\,\delta t\,(1-K_2\,\delta t)^{n-n_o-1} \sum_{i-n_o}^{n-1} \left[\frac{1-K_d t}{1-K_2\,\delta t}\right]^{i-n_o}$$

$$= K_1\,\delta t\,(1-K_2\,\delta t)^{n-n_o-1} \cdot \frac{1-\left[\frac{1-K_d\,\delta t}{1-K_2\,\delta t}\right]^{n-n_o}}{1-\left[\frac{1-K_d\,\delta t}{1-K_2\,\delta t}\right]}$$

Collecting terms, one obtains

$$q_n = K_1 \, \delta t \; \frac{(1-K_2 \, \delta t)^{n-n_o} - (1-K_d \, \delta t)^{n-n_o}}{1-K_2 \delta t - (1-K_d \delta t)}$$

$$= \frac{K_1}{K_d-K_2} \left[(1-K_2 \, \delta t)^{n-n_o} - (1-K_d \, \delta t)^{n-n_o} \right] . \qquad (4.71)$$

Recall that $\delta t = (t-t_o)/(n-n_o)$. Taking the limit as $\delta t \to 0$ yields

$$\lim_{n \to \infty} q_n = \frac{K_1}{K_2-K_d} \left[e^{-K_d(t-t_o)} - e^{-K_2(t-t_o)} \right] . \qquad (4.72)$$

Therefore

$$\int_{\infty}^{\infty} g(x,t \mid t_o) = \frac{K_1}{K_2-K_d} \left[e^{-K_d(t-t_o)} - e^{-K_2(t-t_o)} \right] . \qquad (4.73)$$

q.e.d.

Making the variable transformation $t \to t-t_o$, the initial conditions for Equation (4.39) become

1. $$\int_{-\infty}^{\infty} g(x,t)\,dx = \frac{K_1}{K_2-K_d}\left[e^{-K_d t} - e^{-K_2 t}\right] , \qquad (4.74)$$

2. $$\lim_{t \to 0^+} g(x,t) = 0 , \qquad (4.75)$$

3. $$\lim_{x \to 0^-} g(x,t) = \lim_{x \to 0^+} g(x,t) , \qquad (4.76)$$

4. $$\lim_{x \to -\infty} g(x,t) = \lim_{x \to \infty} g(x,t) = 0 . \qquad (4.77)$$

Restating Equation (4.39), one has

$$\frac{\partial g(x,t)}{\partial t} = E \frac{\partial^2 g(x,t)}{\partial x^2} - K_2\, g(x,t) + K_1\, f(x,t)$$

Let $\tau(x,s)$ be the Laplace transform of $g(x,t)$,

i.e.,

$$\tau(x,s) = \int_0^{\infty} g(x,t)\ e^{-st}\ dt. \qquad (4.78)$$

Taking the Laplace transform of Equation (4.39) gives

$$s\ \tau(x,s) - g(x,0^+) = E\ \frac{\partial^2 \tau(x,s)}{\partial x^2} - K_2 \tau(x,s) + K_1 \pi(x,s), \qquad (4.79)$$

where $\pi(x,s)$ is the Laplace transform of $f(x,t)$. From initial condition (4.75), we see that

$$\frac{\partial^2 \tau(x,s)}{\partial x^2} - \frac{(s+K_2)}{E}\ \tau(x,s) = -K_1 \pi(x,s) . \qquad (4.80)$$

First, consider the homogeneous equation

$$\frac{\partial^2 \tau(x,s)}{\partial x^2} - \frac{(s+K_2)}{E}\ \tau(x,s) = 0 . \qquad (4.81)$$

The general solution to Equation (4.81) is

$$\tau h\,(x,s) = c_1\, e^{m_1 x} + c_2 e^{m_2 x} \quad , \qquad (4.82)$$

where

$$m_1 = \sqrt{\frac{s+K_2}{E}} \quad , \quad m_2 = -\sqrt{\frac{s+K_2}{E}}$$

and c_1 and c_2 are constants.

It is now necessary to find any particular solution to Equation (4.80). Suppose there exists a particular solution of the form

$$Y = A\, e^{\pm \sqrt{\frac{s+K_d}{E}}\; x} \quad , \qquad (4.83)$$

where A is a constant. Substituting into Equation (4.80) and solving for A one obtains

$$A = \frac{K_1}{K_2 - K_d}\,(4E(s+K_d))^{-1/2}. \qquad (4.84)$$

Thus, a general solution to Equation (4.80) is

$$\tau\,(x,s) = c_1\, e^{m_1 x} + c_2\, e^{m_2 x} + Y \ . \qquad (4.85)$$

Using initial conditions (4.76) and (4.77) one obtains

$$\tau(x,s) = \begin{cases} c\ e^{m_1 x} + Y \text{ when } x \leqslant 0 , \\ \\ c\ e^{m_2 x} + Y \text{ when } x > 0 . \end{cases} \tag{4.86}$$

Taking the Laplace transform of both sides of initial condition (4.74) yields

$$\int_{-\infty}^{\infty} \tau(x,s)\ dx = \frac{K_1}{K_2-K_d}\left[\frac{1}{s+K_d} - \frac{1}{s+K_2}\right] . \tag{4.87}$$

Integrating the value of $\pi(x,s)$ given in Equation (4.86) yields

$$\int_{-\infty}^{\infty} \tau(x,s)\ dx = 2c\sqrt{\frac{E}{s+K_2}} + \frac{K_1}{(K_2-K_d)(s+K_d)} . \tag{4.88}$$

Thus

$$c = \frac{-K_1}{2\sqrt{E(s+K_2)}\,(K_2-K_d)} , \tag{4.89}$$

which gives

$$\tau\,(x,s) \;=\; \frac{K_1}{2\,(K_2\text{-}K_d)\;\sqrt{E}}\;\left[\,(s+K_d)^{1/2}\;e^{+x\sqrt{\frac{s+K_d}{E}}}\right.$$

$$\left. -\;(s+K_2)^{1/2}\,e^{+x\sqrt{\frac{s+K_2}{E}}}\;\right]\;\text{if}\;\;x \leqslant 0\;,$$

and

$$\tau\,(x,s) \;=\; \frac{K_1}{2(K_2\text{-}K_d)\;\;E}\;\left[\,(s+K_d)^{\;1/2}\;\;e^{-x\sqrt{\frac{s+K_d}{E}}}\right.$$

$$\left. -\,(s+K_2)^{1/2}\;e^{-x\sqrt{\frac{s+K_2}{E}}}\;\;\right]\;\;\text{is}\;\;x > 0\;. \qquad (4.90)$$

Inverting $\pi(x.s)$ yields

$$g\,(x,t) \;=\; \frac{K_1}{2(K_2\text{-}K_d)\;\sqrt{\pi E t}}\;\;e^{\frac{-x^2}{4Et}}\;\;\left[e^{-K_d t}\,\text{-}\,e^{-K_2 t}\,\right]\;. \qquad (4.91)$$

Taking the inverse transformation $t \to t\text{-}t_o$, the solution to Equation (4.39) is

$$g\,(x,t\mid t_o) = \frac{K_1}{2(K_2-K_d)\sqrt{\pi E(t-t_o)}}\; e^{\frac{-x^2}{4E(t-t_o)}} \left[e^{-K_d(t-t_o)}-e^{-K_2(t-t_o)}\right] . \tag{4.92}$$

The solution to Equation (4.22) is found by translating Equation (4.92) on the x axis by a distance

$$-\int_{t_o}^{t} U\,(\tau)\; d\tau \; .$$

Using the results derived above, $f(x,t|t_o)$ and $g(x,t|t_o)$ can be found rapidly for any value of t, by use of the computer. Thus, the time barrier encountered in the difference equation solution is removed. It should be pointed out that this method yields approximations to $u(m,n|n_o)$ and $v(m,n|n_o)$ since δt has a physical value and, despite the suggestiveness of the notation, does not approach 0.

The approximations of $u(n,m|n_o)$ and $v(m,n|n_o)$ for any $t = n\delta t$ are found by evaluating $f(x,t|t_o)$ and $g(x,t|t_o)$, respectively, and multiplying by $2\delta x$.

The intent of Figures 29 through 36 is to illustrate the degree of accuracy obtained by using the discrete probability mass functions $f(x,t)\ 2\delta x$ and $g(x,t)\ 2\delta x$ as approximations to the probability mass functions $u(x,t)$ and $v(x,t)$, respectively. The ordinates of the difference equation solutions are denoted by the symbol X. For clarity, the ordinates of the approximating mass functions have been joined by a curve. It should be pointed out that the probabilities represented by the continuous curve are in fact discrete. The curve is drawn only to facilitate comparison.

δt for this set of plots is .0025 days. One would expect better agreement for smaller δt and poorer results if δt is increased. All of the comparisons are for small lapses of time, since it was not practical to solve the difference equations for large time lapses. This is, of course, a disadvantage since we are interested in large values of time. However, it may be noted that the comparisons of the plots do not appear to be affected by the increases in time shown.

Continuous Source

The analysis in Continuous Approximations to Difference Equations gives the probability of a unit of BOD or OD released at t_o being at location x after time t. It will be recalled that δt was the time required for an amount of pollution equal to Δ to be released. A continuous release of pollution can be simulated by releasing an increment of pollution Δ, at each interval of time δt. The first release will be at $t_o = 0$, and the last at $t_o = t$, where t is the time of interest. The probability of one of these releases being at location x after time t was obtained in the preceding chapter. Hence, the probability of a given concentration at location x after time t is the convolution of the individual probabilities.

The convolution can be derived by considering the situation as a set of individual Bernoulli trials. A given unit being at x after time t will be considered a success for that trial, i.e., the probability of success for the release at step n_i is $u(m,n|n_i)$, where $u(m,n|n_i)$ is as in Difference Equations, and $m = x/\delta x$, $n = t/\delta t$.

Let $p_k(m,n) = \Pr[\text{k BOD units at location } m\delta x \text{ after time } n\delta t]$, and $\underset{\sim}{p}_k(n,m) = \Pr[\text{OD concentration is } k\Delta \text{ at location } m\delta x \text{ after time } n\delta t]$.

Let

$$P(m,n,s) = \sum_{k=0}^{\infty} p_k(m,n)\, s^k \quad , \tag{4.93}$$

and

$$\underset{\sim}{P}(m,n,s) = \sum_{k=0}^{\infty} \underset{\sim}{p}_k(m,n)\, s^k \quad . \tag{4.94}$$

Then

$$P(m,n,s) = \prod_{i=0}^{n} [(1-u(m,n \mid i)) + u(m,n \mid i)\, s] . \qquad (4.95)$$

and

$$\underset{-}{P}(m,n,s) = \prod_{i=0}^{n} [(1-v(m,n \mid i)) + v(m,n \mid i)\, s] , \qquad (4.96)$$

The probability of the system being in any particular state can be found by picking out the desired coefficients in P(m,n,s) and $\underset{-}{P}$(m,n,s).

A Fortran IV program has been written which performs the multiplication and gives the complete distribution for any point in time and space. However, the time required for the multiplication is large if n is large.

Some information can be obtained from the generating function without resorting to binomial multiplication. In particular, the mean and variance of a distribution with generating function F(s) are given by

$$E(k) = F'(s) \Big|_{s=1} \qquad (4.97)$$

and

$$Var(k) = F''(s) \Big|_{s=1} + F'(s) \Big|_{s=1} - (F'(s))^2 \Big|_{s=1} . \qquad (4.98)$$

Now

$$P'(m,n,s) = \sum_{j=0}^{n} u(m,n\,|\,j) \prod_{\substack{i=0\\ i\neq j}}^{n} \left[(1-u(m,n\,|\,i)) + u(m,n\,|\,i)\,s \right] , \tag{4.99}$$

and

$$P''(m,n,s) = \sum_{j=0}^{n} u(m,n\,|\,j) \sum_{\substack{k=0\\ k\neq j}}^{n} u(m,n\,|\,k)$$

$$\cdot \prod_{\substack{i=0\\ i\neq j\\ i\neq k}} \left[(1-u(m,n\,|\,i)) + u(m,n\,|\,i)\,s \right] . \tag{4.100}$$

Therefore,

$$\text{Mean (BOD state)} = \sum_{j=0}^{n} u(m,n\,|\,j) , \tag{4.101}$$

and

$$\text{Var (BOD state)} = \sum_{j=0}^{n} u(m,n \mid j) - \sum_{j=0}^{n} (u(m,n \mid j))^2 \quad . \qquad (4.102)$$

Similarly,

$$\text{Mean (OD state} = \sum_{j=0}^{n} v(m,n \mid j) \quad , \qquad (4.103)$$

and

$$\text{Var (OD state)} = \sum_{j=0}^{n} v(m,n \mid j) - \sum_{j=0}^{n} (v(m,n \mid j))^2 \quad . \qquad (4.104)$$

The moments as given in Equations (4.101) through (4.104), can be calculated comparatively rapidly when $u(m,n|j)$ and $v(m,n|j)$ are approximated by the methods in Continuous Approximations to Difference Equations.

Continuous Approximations to Distributions

By again using the technique of letting δt approach 0 the calculations may be greatly facilitated. First, analytic forms for the distributions of BOD and OD will be obtained. Secondly, the evaluations of the mean and variance can be accomplished in a fraction of the time needed to perform the summations in Equations (4.101) through (4.104).

It will be recalled that $u(m,n|i)$ and $v(m,n|i)$ were approximated as follows:

$$u(m,n \mid i) \approx f(x,t \mid t_i)\ 2\delta x, \qquad (4.105)$$

and

$$v(m,n \mid i) \approx g(x,t \mid t_i)\ 2\delta x\ , \qquad (4.106)$$

where f and g are defined in Continuous Approximations to Difference Equations. Also the increase in the BOD concentration at the source in time δt is Δ. Thus,

$$\Delta = \frac{W\ \delta t}{A\ \delta x}\ , \qquad (4.107)$$

where W is the rate at which pollution enters the estuary, and A is the cross-sectional area. Therefore,

$$u(m,n \mid i) \approx \frac{2W}{A\Delta} f(x,t \mid t_i)\ \delta t\ , \qquad (4.108)$$

and

$$v(m,n \mid i) \approx \frac{2W}{A\Delta} g(x,t \mid t_i)\ \delta t\ . \qquad (4.109)$$

B. O. Koopman (39) gives necessary and sufficient conditions for a set of Bernoulli trails (with probabilities of success not necessarily equal) to converge to a Possion distribution.

Let P(k) be the probability of exactly k successes in a Poisson distribution with mean m. Let $P_n(k)$ be the probability of exactly k successes in n independent Bernoulli trials. The mean of $P_n(k)$ will be represented by $m(n)$.

Koopman's theorem states that a necessary and sufficient condition for

$$\lim_{n\to\infty} P_n(k) = P(k)' \tag{4.110}$$

is that both the following equations hold:

a. $$\lim_{n\to\infty} m(n) = m \ , \tag{4.111}$$

b. $$\lim_{n\to\infty} \max_{1\leqslant i\leqslant n} p_{n,i} = 0 \ , \tag{4.112}$$

where $p_{n,i}$ is the probability of success on the i^{th} trial in the n^{th} set.

In our analysis it is seen that as δt approaches zero the number of Bernoulli trials approaches infinity. Equations (4.108) and (4.109) show that condition (b.) of the theorem is satisfied. Thus, if the means as given in Equations (4.101) and (4.103) exist, and as δt approaches zero, approach limits m_1 and m_2, respectively, the BOD and OD distributions will approach Poisson distributions with parameters m_1 and m_2, respectively.

Substituting Equation (4.108) into Equation (4.101) gives

$$\text{Mean (BOD state)} \approx \frac{2W}{A\Delta} \sum_{i=0}^{n} f(x,t \mid t_i)\ \delta t \ . \tag{4.113}$$

Therefore, as $\delta t \to 0$,

$$\text{Mean (BOD state)} \to \frac{2W}{A\Delta} \int_0^t f(x,t \mid \tau)\ d\tau \ . \tag{4.114}$$

Hence, the distribution of the BOD state approaches a Poisson distribution with parameter

$$m_1 = \frac{2W}{A\Delta} \int_0^t f(x,t\,|\,\tau)\, d\tau \ . \tag{4.115}$$

Similarly, the distribution of the OD state approaches a Poisson distribution with parameter

$$m_2 = \frac{2W}{A\Delta} \int_0^t g(x,t\,|\,\tau)\, d\tau \ . \tag{4.116}$$

Substituting Equation (4.108) into Equation (4.102) yields

$$\mathrm{Var\,(BOD\ state)} \approx \frac{2W}{A\Delta} \sum_{i=0}^{n} f(x,t\,|\,t_i)\ \delta t$$

$$- \left(\frac{2W}{A\Delta}\right)^2 \sum_{i=0}^{n} [f(x,t\,|\,t_i)]^2\ \delta t^2 \ . \tag{4.117}$$

As $\delta t \to 0$ the first term is m_1 and the second term is 0. Thus, the mean and variance are equal; as must be the case for the Poisson distribution.

The means of the distributions may be obtained by evaluating the integrals in Equations (4.115) and (4.116). This will prove to be more efficient than performing the summations in Equations (4.101) and (4.103).

Figures 37 through 44 compare the Poisson approximation to the distribution found by performing the multiplications in Equations (4.95) and (4.96). The discrete plots represent the solution to Equations (4.95) and

(4.96). The continuous curves represent the Poisson distributions with means obtained by Equations (4.101) and (4.103). As in the earlier set of plots, the continuous curve actually represents a discrete distribution; the continuous representation is for visual convenience.

δt is .0025 days in Equations (4.95) and (4.96). Because of limitations on computer time, comparisons are only for small values of time.

The level of pollution is Δ times the state. Therefore, if $\mu(x,t)$ and $\sigma^2(x,t)$ are the BOD mean and variance, respectively, at location x and time t, one has

$$\mu(x,t) = \Delta m_1 = \frac{2W}{A} \int_0^t f(x,t \mid \tau)\, d\tau \quad , \tag{4.118}$$

and

$$\sigma^2(x,t) = \Delta^2 m_1 = \Delta \mu(x,t) \quad . \tag{4.119}$$

Similarly, if $\mu(x,t)$ and $\sigma^2(x,t)$ represent the mean and variance for OD, one obtains

$$\underset{\sim}{\mu}(x,t) = \Delta m_1 = \frac{2W}{A} \int_0^t g(x,t \mid \tau)\, d\tau \, , \tag{4.120}$$

and

$$\sigma^2(x,t) = \Delta^2 m_2 = \Delta \underset{\sim}{\mu}(x,t) \ . \qquad (4.121)$$

It is seen that the mean concentrations of BOD and OD are independent of Δ; and the variances are directly proportional to Δ. It is also interesting to note that the parameters δt and δx need not be evaluated in order to obtain the distributions.

Variable Source

The proceding analysis assumes that the pollution begins entering the estuary at time 0 and continues at a constant rate up to the time of interest. The work can easily be generalized to the situation where the source varies in a certain pre-described manner.

First, consider the situation of a transient source. The source begins emitting pollution at time s_1 at a constant rate W_1, and ceases emitting at time $s_2 \leqslant t$. The diffusion mechanism as described in Difference Equations and Continuous Approximations to Difference Equations would be unchanged. However, in this situation the first release is at $t_o = s_1$, and the last at $t_o = s_2$. The analysis of Continuous Source remains intact, except that the number of Bernoulli trials is $(s_2 - s_1)/\delta t$. Thus, Equation (4.115) becomes

$$\text{Mean (BOD) state} = \frac{2W_1}{A\Delta} \int_{s_1}^{s_2} f(x,t \mid \tau)\, d\tau \ . \qquad (4.122)$$

General results can be obtained utilizing Equation (4.122). Suppose the rate at which pollution enters varies discretely, i.e., the rate remains constant for some well defined length of time and then jumps to a new level. This condition can be stated quantatively as a vector

$$W = \underset{n\times 1}{[W_i]} \qquad (4.123)$$

where

W_i = pollution emission rate during time interval s_i to s_{i+1} .

By the principle of superposition the net effect is the convolution of the individual factors given by Equation (4.122). The net result will be a Poisson distribution with parameter equal to the sum of the parameters from the individual factors. Thus,

$$\text{Mean (BOD state)} = \sum_{i=1}^{n} \frac{2W_i}{A\Delta} \int_{s_i}^{s_{i+1}} f(x,t \mid \tau)\, d\tau \ . \qquad (4.124)$$

The mean BOD concentration is

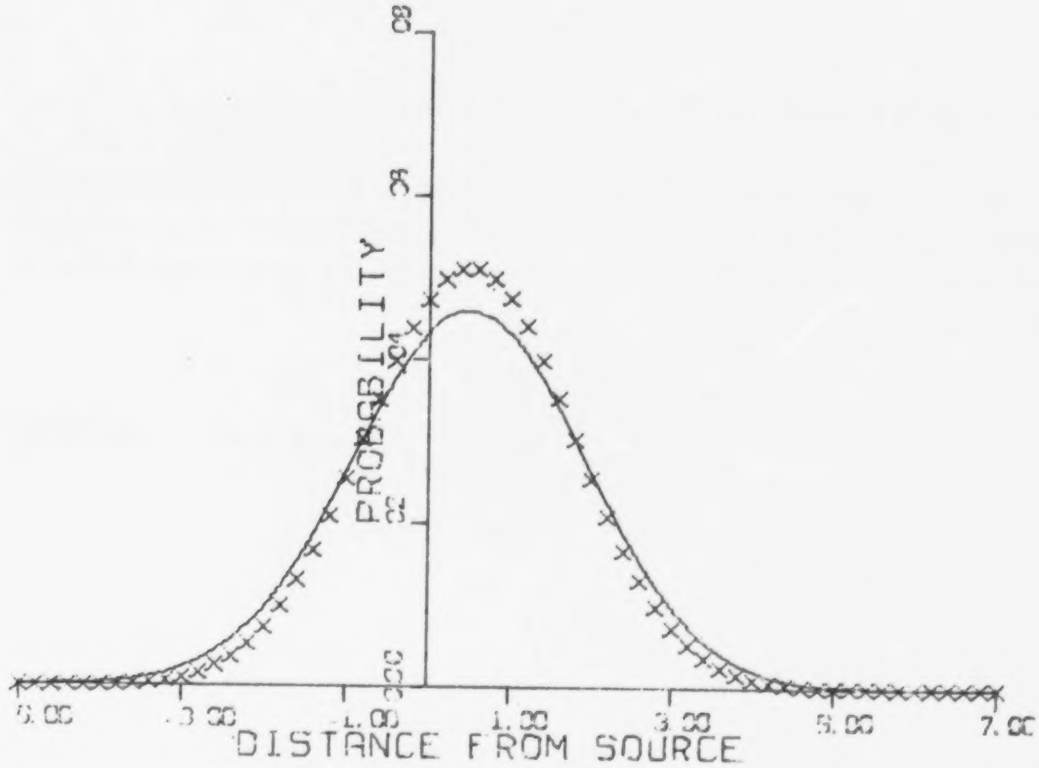
$$\mu\,(x,t) = \sum_{i=1}^{n} \frac{2W_1}{A} \int_{s_i}^{s_{i+1}} f(x,t \mid \tau)\, d\tau \ . \qquad (4.125)$$

The variance is $\Delta\mu(x,t)$.

A similar analysis applied to the OD profile yields

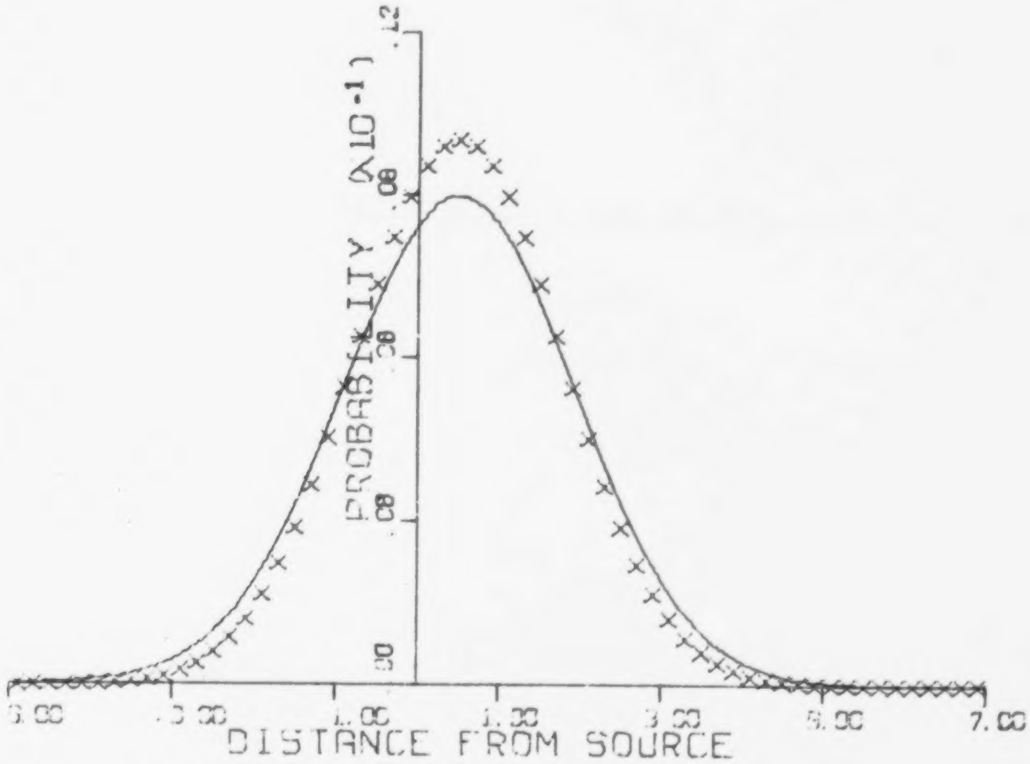
$$\underline{\mu}\,(x,t) = \sum_{i=1}^{n} \frac{2W_1}{A} \int_{s_i}^{s_{i+1}} g(x,t \mid \tau)\, d\tau \ . \qquad (4.126)$$

and the variance is equal to $\Delta\ \underline{\mu}\ (x,t)$.



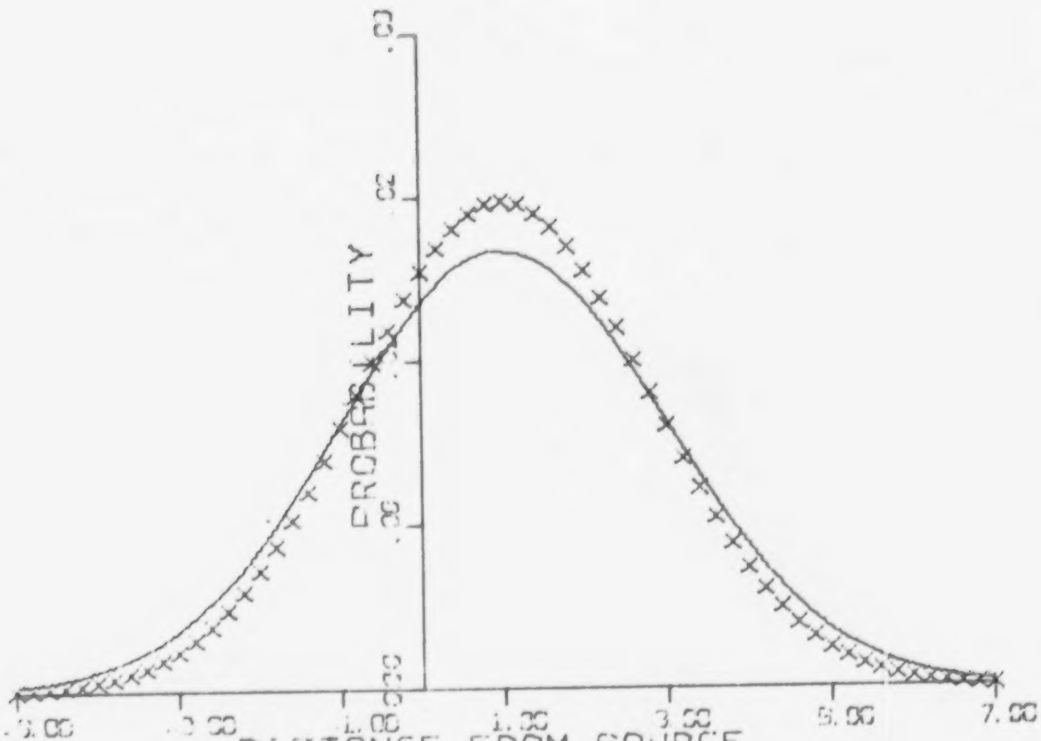
PROBABILITIES FOR BOD PARTICLE
DIFFERENCE EQ. SOLUTIONS VS DIFFERENTIAL EQ. SOLUTIONS
TIME OF TRAVEL IS 05 DAYS

FIGURE 29



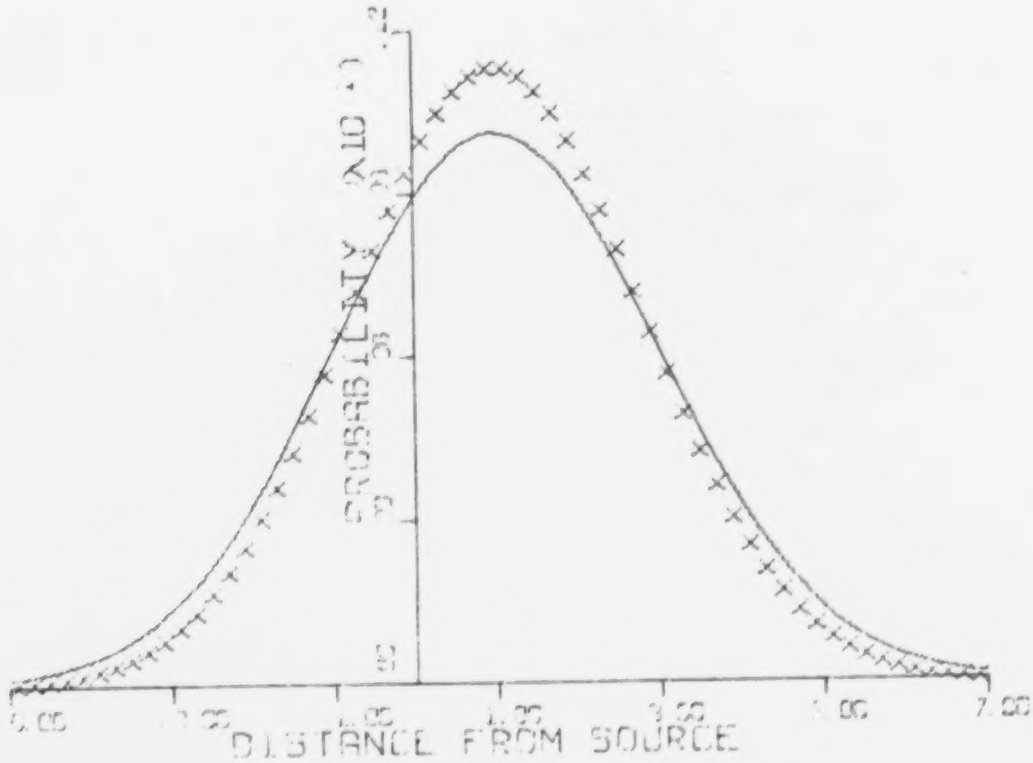
PROBABILITIES FOR OD PARTICLE
DIFFERENCE EQ. SOLUTIONS VS DIFFERENTIAL EQ. SOLUTIONS
TIME OF TRAVEL IS 06 DAYS

FIGURE 30



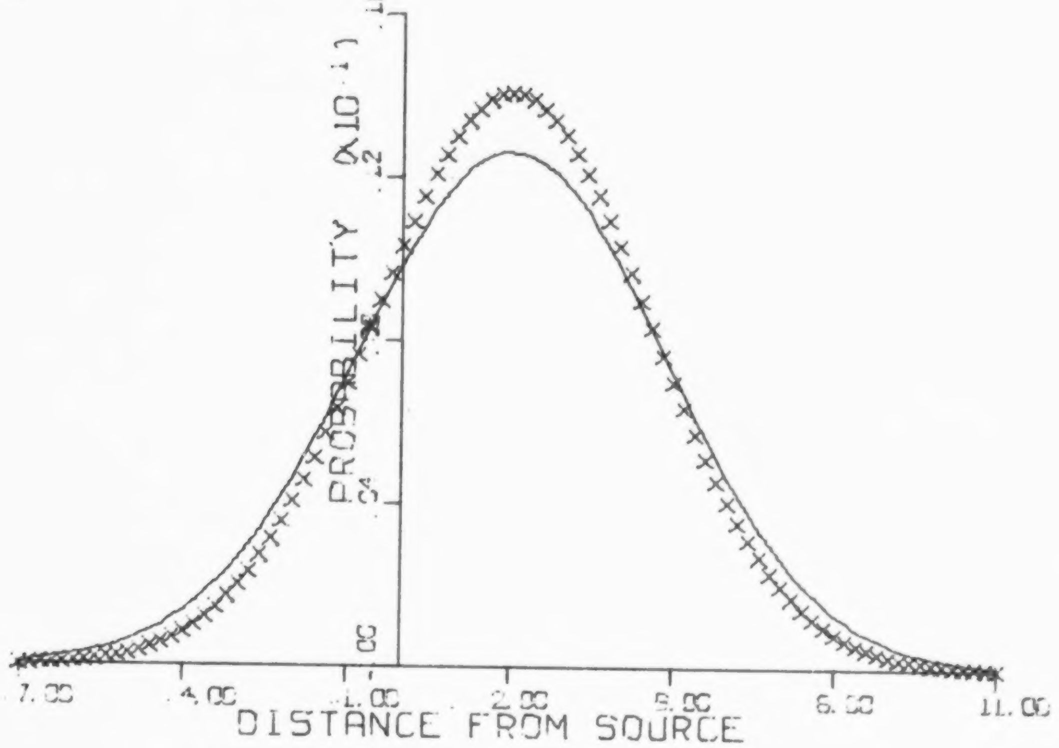
PROBABILITIES FOR 800 PARTICLE
DIFFERENCE EQ. SOLUTIONS VS DIFFERENTIAL EQ. SOLUTIONS
TIME OF TRAVEL IS 1.0 DAY

FIGURE 31



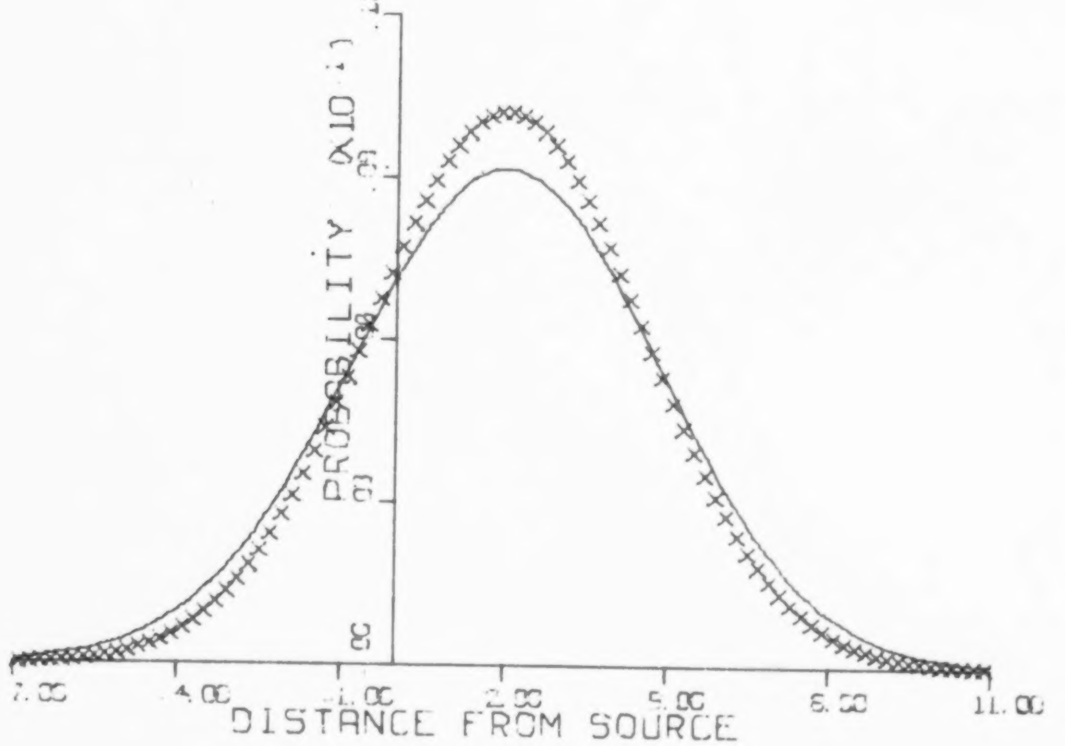
PROBABILITIES FOR 00 PARTICLE
DIFFERENCE EQ. SOLUTIONS VS DIFFERENTIAL EQ. SOLUTIONS
TIME OF TRAVEL IS 1.0 DAY

FIGURE 32



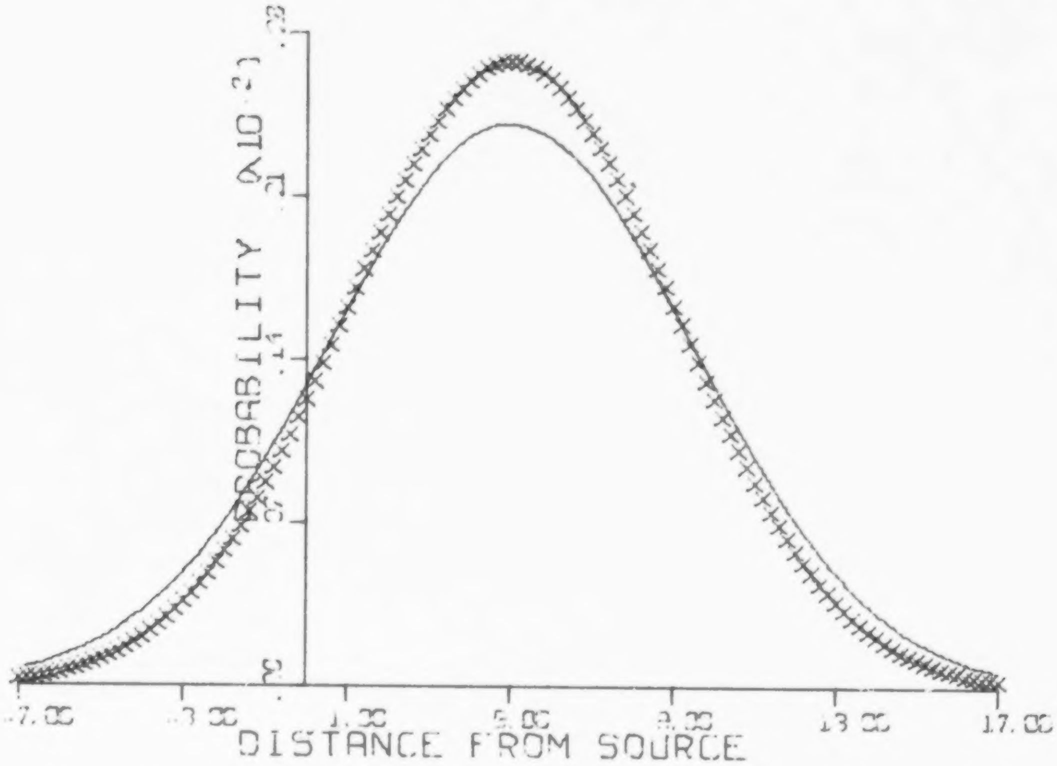
PROBABILITIES FOR 500 PARTICLE
DIFFERENCE EQ SOLUTIONS VS DIFFERENTIAL EQ SOLUTIONS
TIME OF TRAVEL IS 20 DAYS

FIGURE 33



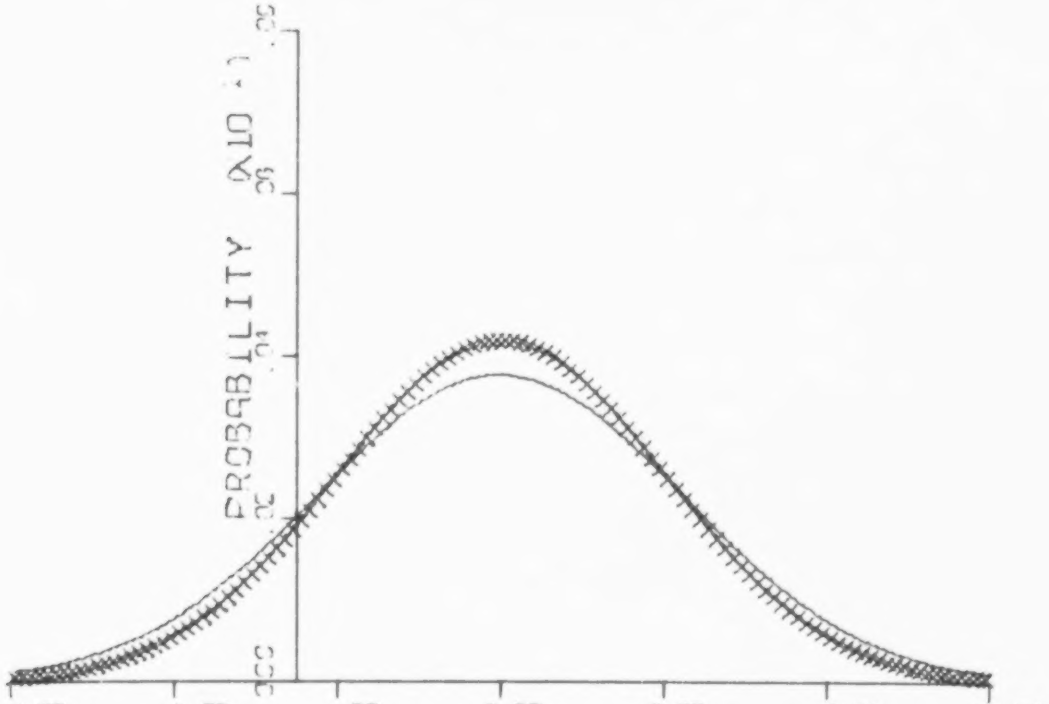
PROBABILITIES FOR 00 PARTICLE
DIFFERENCE EQ SOLUTIONS VS DIFFERENTIAL EQ SOLUTIONS
TIME OF TRAVEL IS 20 DAYS

FIGURE 34



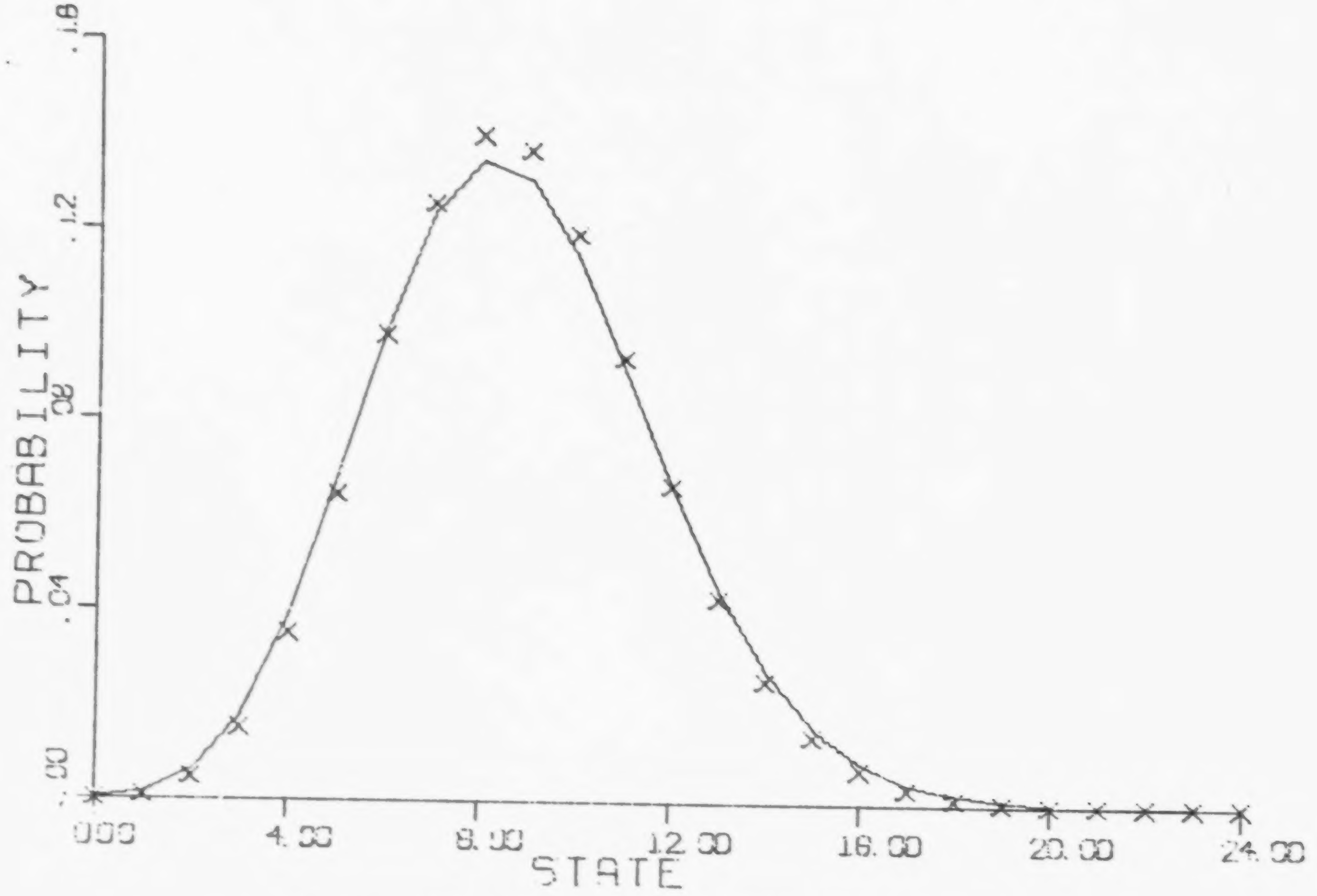
PROBABILITIES FOR 800 PARTICLE
DIFFERENCE EQ SOLUTIONS VS DIFFERENTIAL EQ SOLUTIONS
TIME OF TRAVEL IS 50 DAYS

FIGURE 35



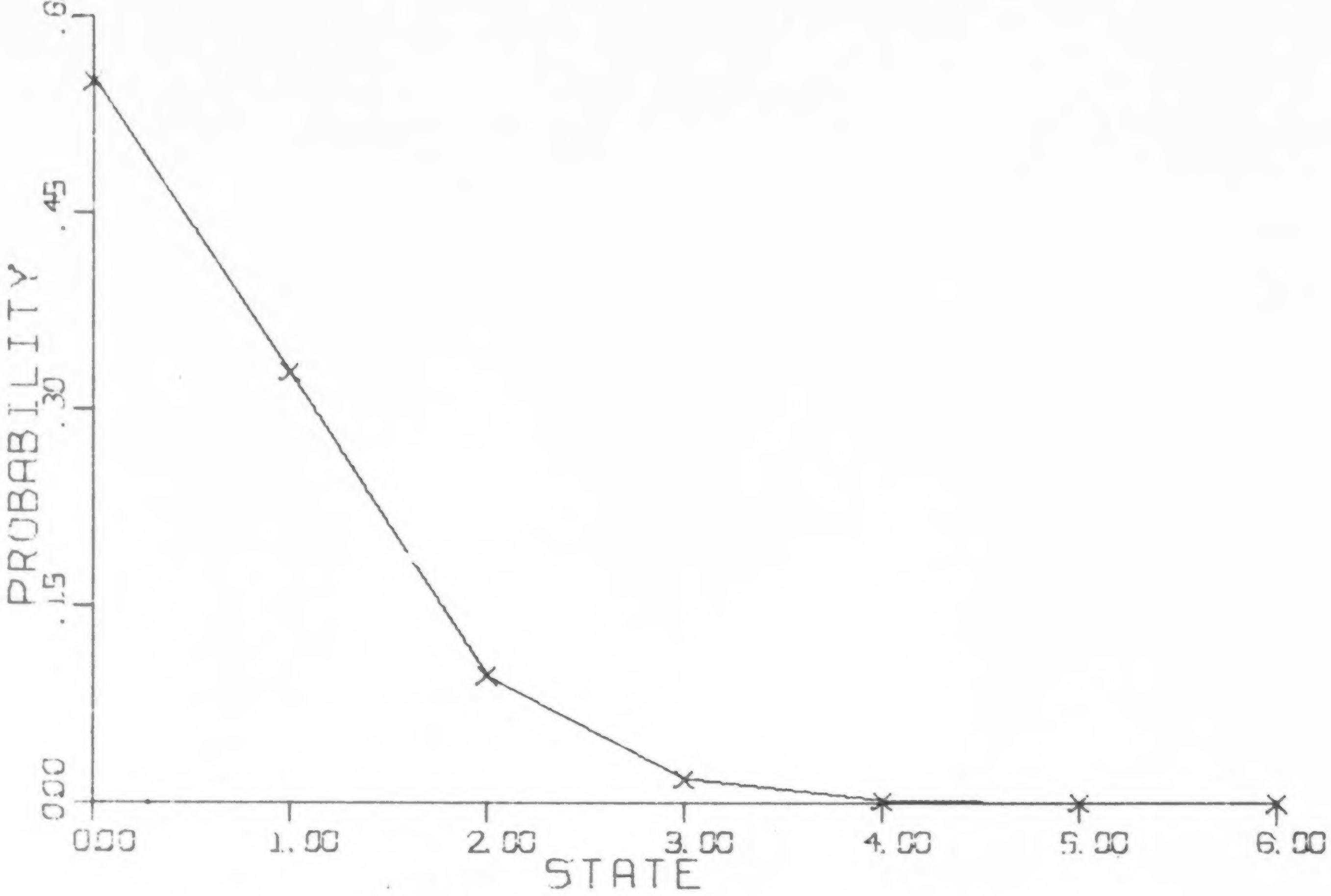
PROBABILITIES FOR 00 PARTICLE
DIFFERENCE EQ SOLUTIONS VS DIFFERENTIAL EQ SOLUTIONS
TIME OF TRAVEL IS 50 DAYS

FIGURE 36



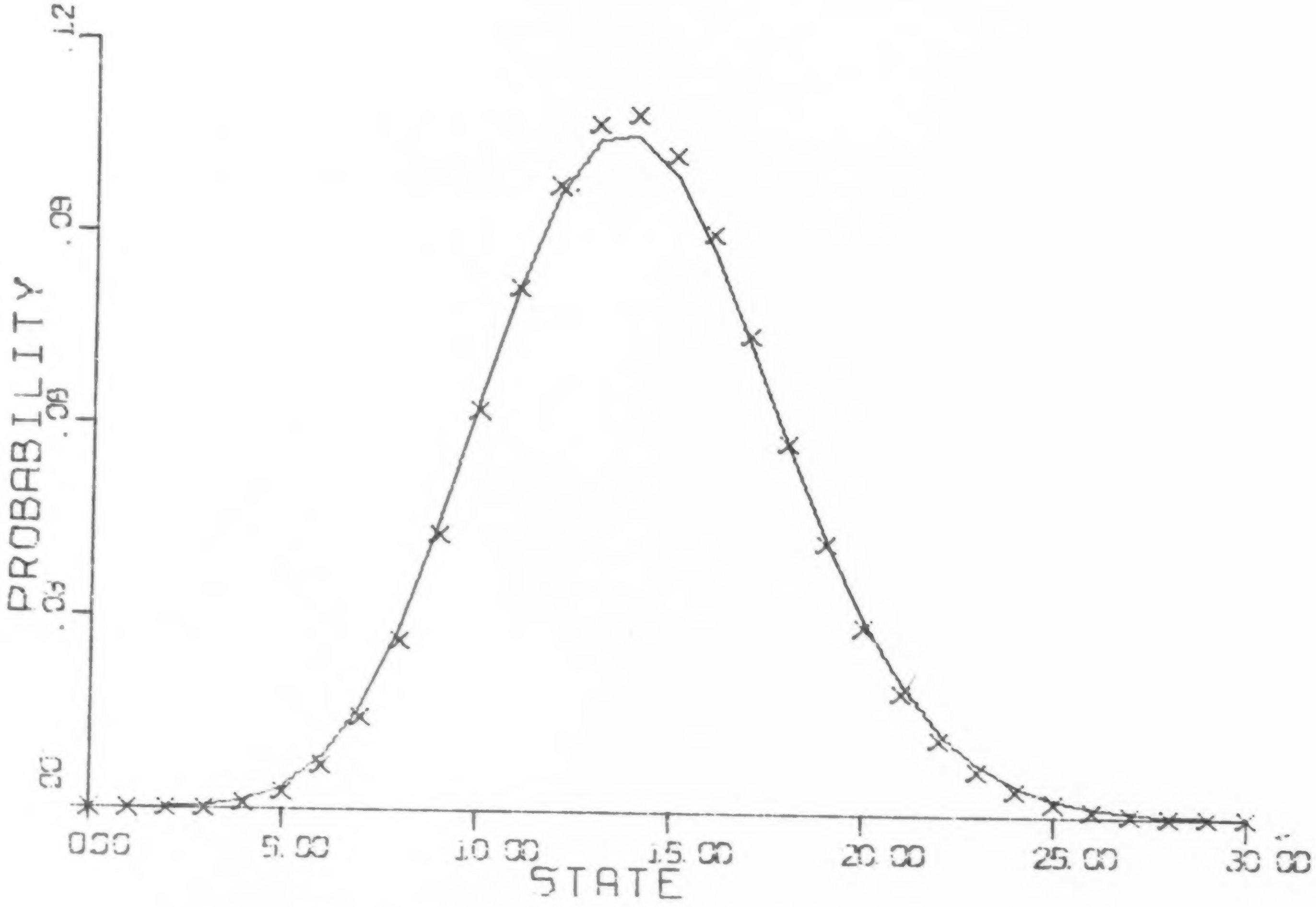
PROBABILITY DISTRIBUTIONS FOR BOD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 05 DAYS

FIGURE 37



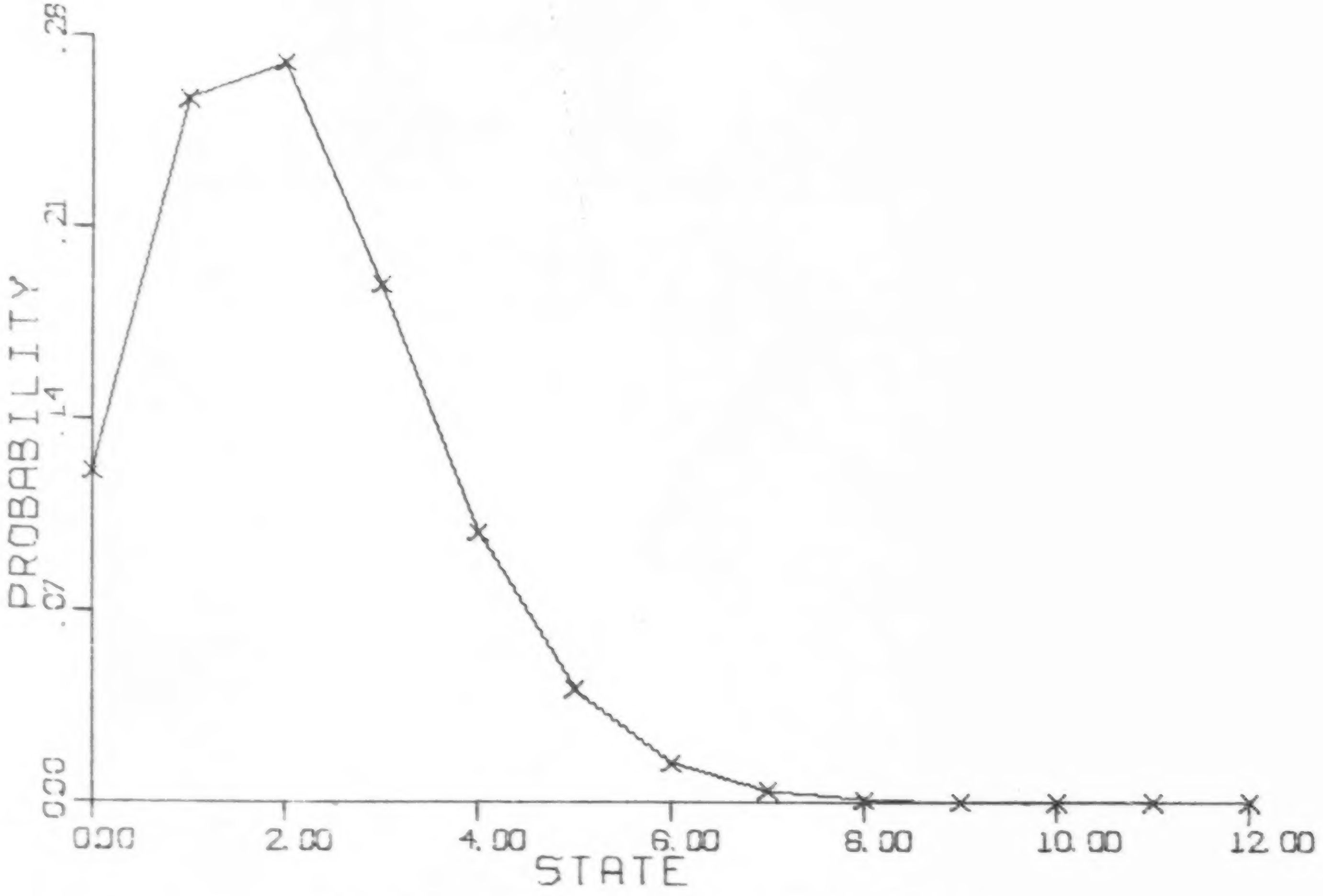
PROBABILITY DISTRIBUTIONS FOR OD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 0.5 DAYS

FIGURE 38



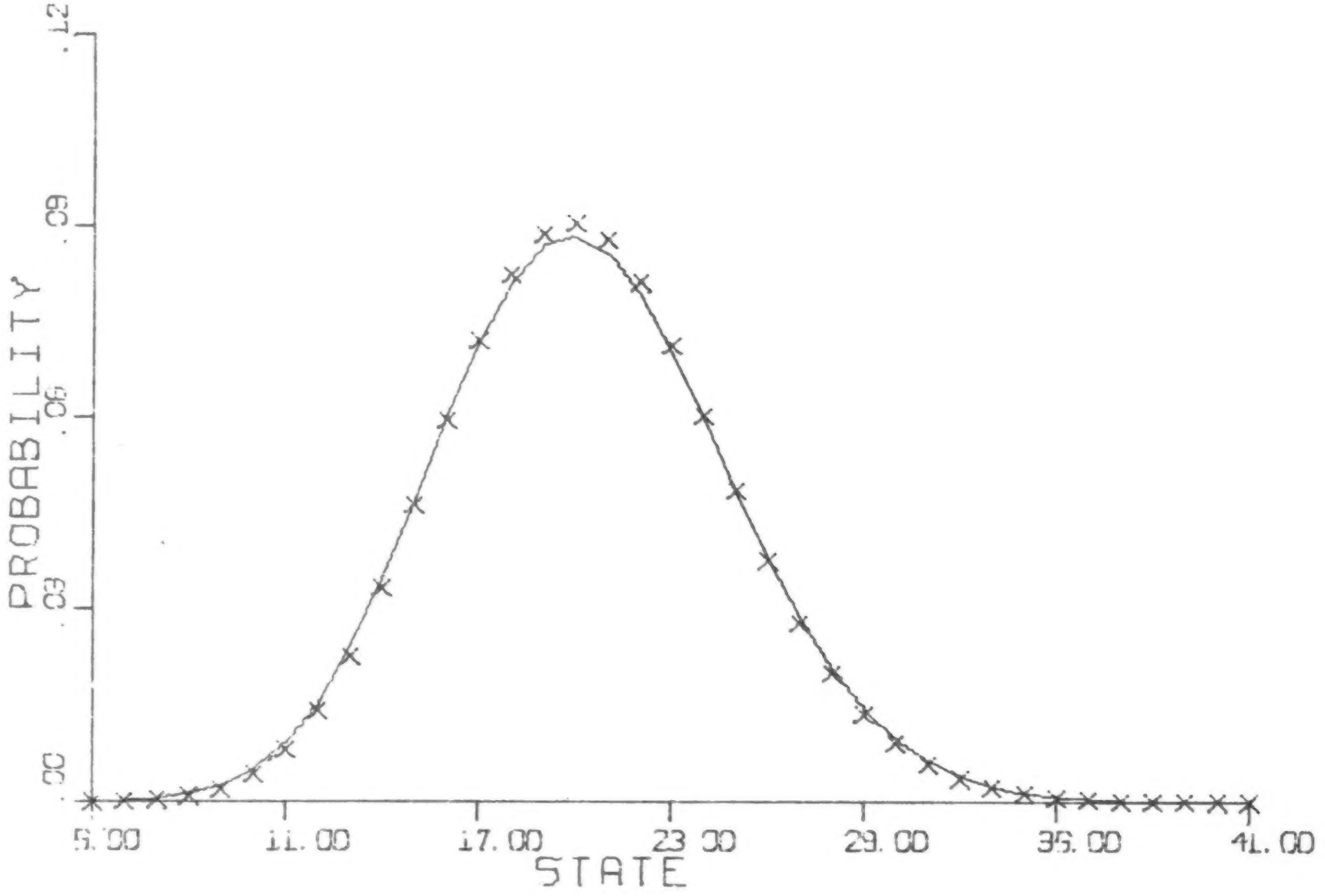
PROBABILITY DISTRIBUTIONS FOR BOD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 10 DAY

FIGURE 39



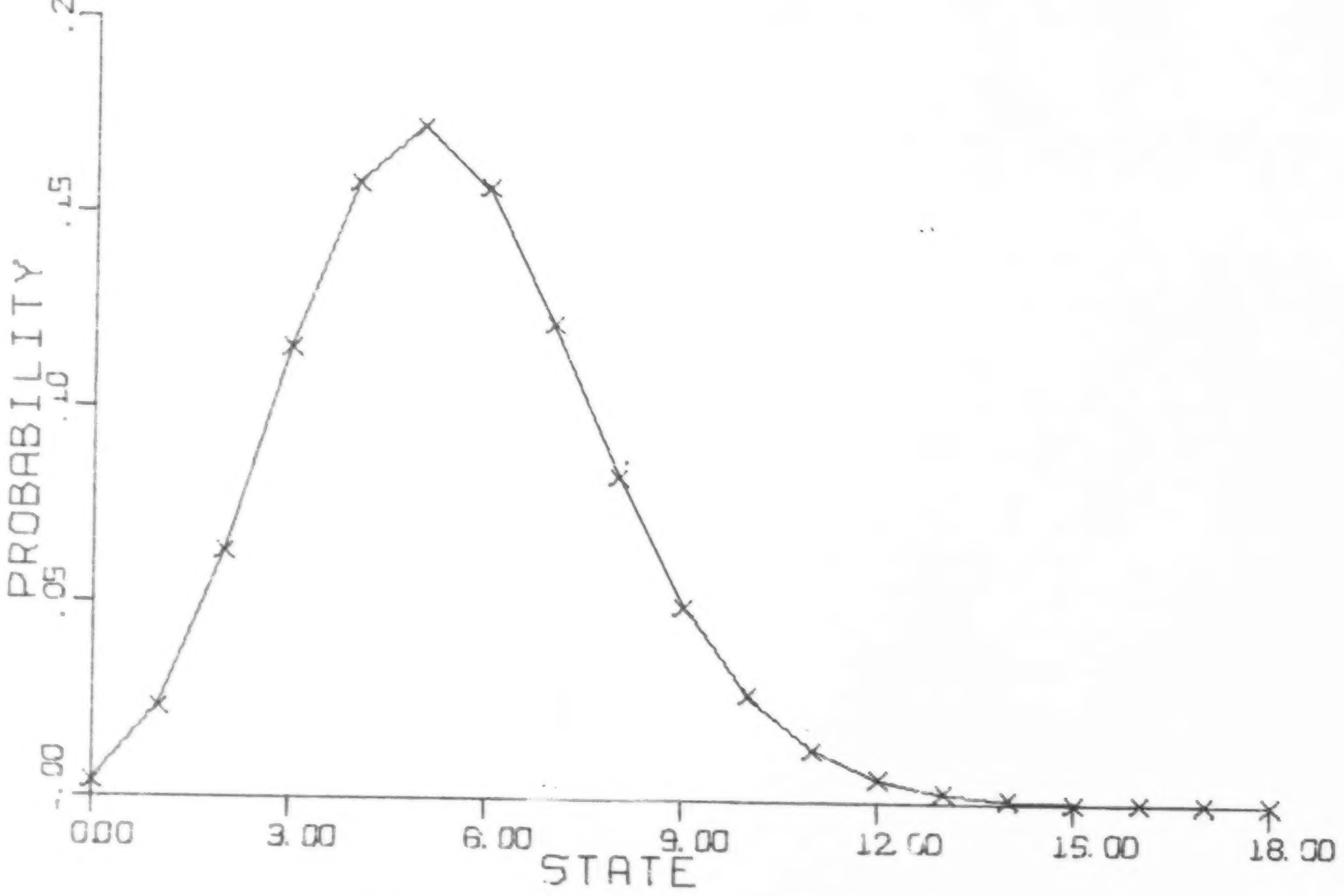
PROBABILITY DISTRIBUTIONS FOR OD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 1.0 DAY

FIGURE 40



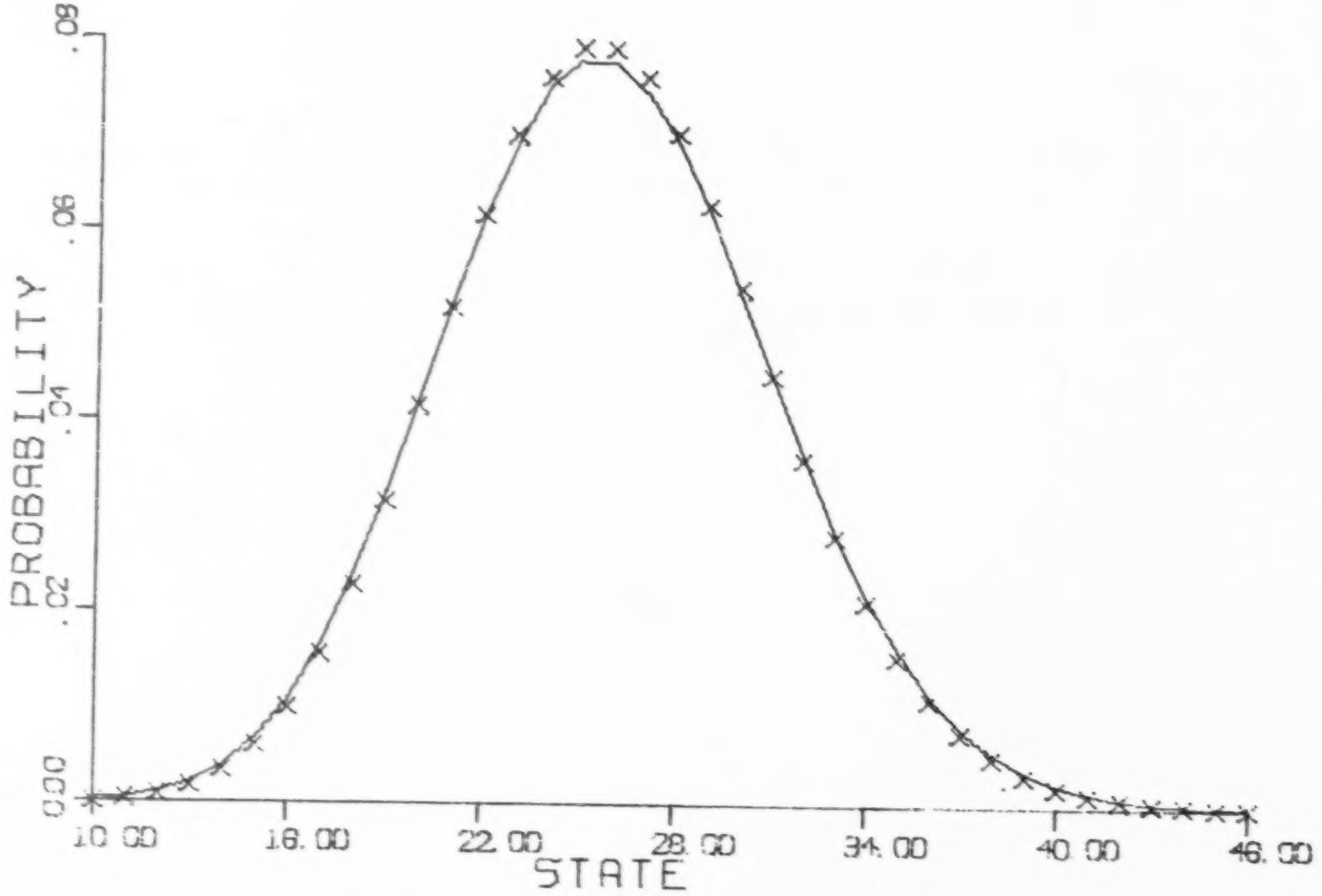
PROBABILITY DISTRIBUTIONS FOR BOD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 20 DAYS

FIGURE 41



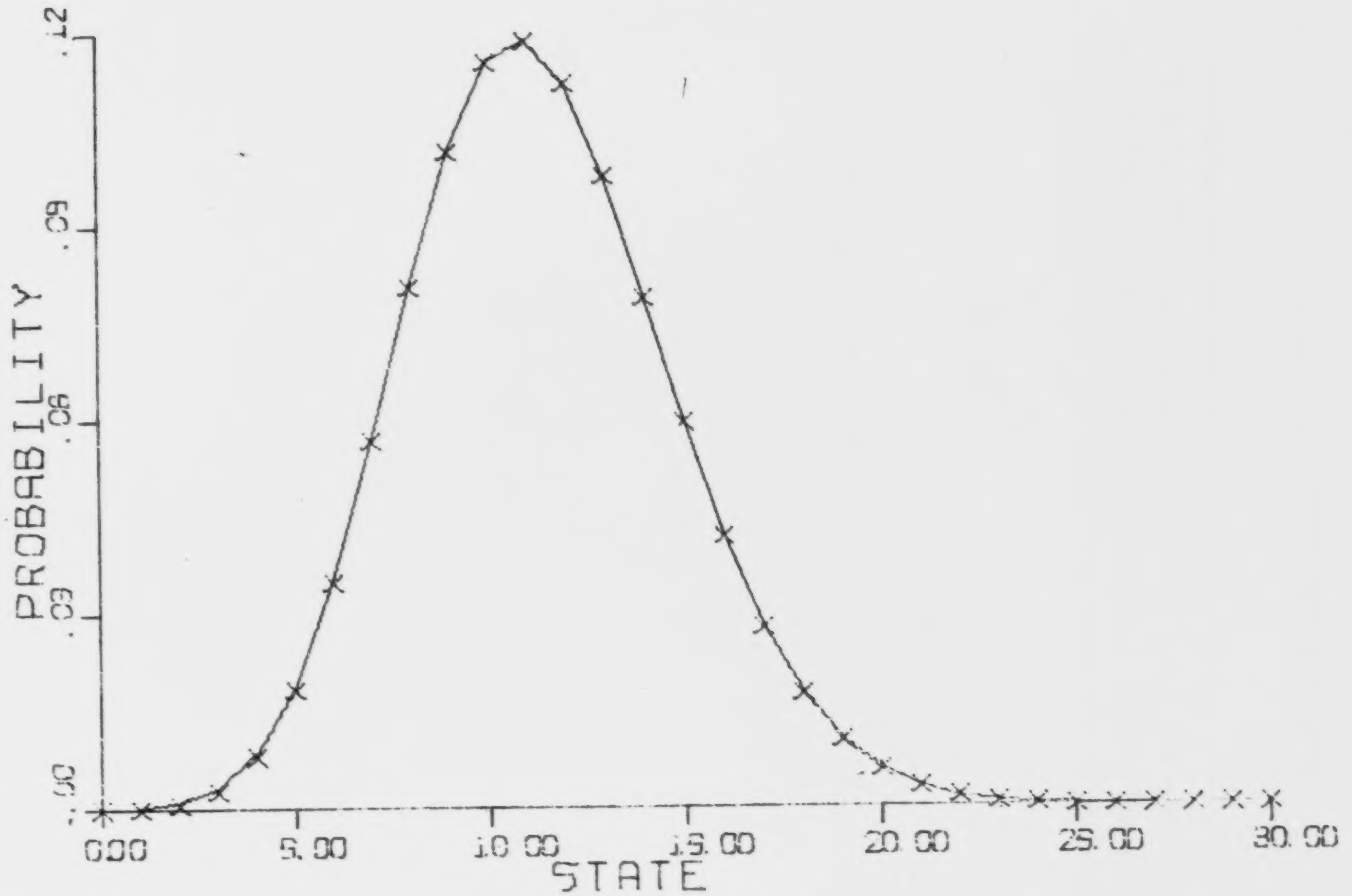
PROBABILITY DISTRIBUTIONS FOR OD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 20 DAYS

FIGURE 42



PROBABILITY DISTRIBUTIONS FOR BOD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 50 DAYS

FIGURE 43



PROBABILITY DISTRIBUTIONS FOR OD
GENERATING FUCTIONS SOLUTION VS POISSON DISTRIBUTION
TIME OF TRAVEL IS 50 DAYS

FIGURE 44

METHOD OF INTEGRATION

In light of the analysis in Diffusion Model, the BOD and OD distributions for any point in time and space can be found by evaluating the integral in Equation (4.118) and (4.120). Substituting the values of $f(x,t|t_o)$ and $g(x,t|t_o)$ given in Equations (4.68) and (4.92) into Equations (4.118) and (4.120) yield

$$\mu(x,t) = \frac{W}{A\sqrt{\pi E}} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-(x-\int_\tau^t u(\xi)\,d\xi)^2/(4E(t-\tau))}$$

$$e^{-K_d(t-\tau)}\,d\tau \ , \tag{5.1}$$

and,

$$\underset{\sim}{\mu}(x,t) = \frac{W}{A\sqrt{\pi E}} \frac{K_1}{K_2-K_d} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-(x-\int_\tau^t u(\xi)d\xi)^2/(4E(t-\tau))}$$

$$[\ e^{-K_d(t-\tau)} - e^{-K_2(t-\tau)}\ ]\ d\tau \ . \tag{5.2}$$

It will be necessary to obtain a great many distributions for different values of x and t in order to obtain a complete description of the estuary. Hence, some effort is justified in finding the most efficient method of evaluating Equations (5.1) and (5.2). It is clear that some form of numerical integration will be required.

Four numerical methods of integration were compared.

1. Simpson's Rule: This involves evaluating the functions at N equally spaced intervals. Evaluation of the integrals is done by means of Simpson's Rule together with Newton's 3/8 Rule.

2. Trapezoidal Rule with equally spaced intervals: Again the functions are evaluated at N equally spaced intervals and the integrals are evaluated by means of the Trapezoidal Rule.

3. Hermitian Formula: At N equally spaced intervals the function, its first and second derivatives are evaluated. The integrals are evaluated by means of Hermitian Sixth Order Integrating Formula.

4. Trapezoidal Rule with increasing intervals: For this method, Equations (5.1) and (5.2) are transformed by the change of variables $s = t-\tau$. The functions are evaluated at the N points $s(k)$, $k = 1 \ldots N$. The $s(k)$'s are defined as follows:

$$s(1) = -\frac{1}{K_d} \left\{ \ln \left[1-(1-e^{-K_d t})/N \right] \right\} ,$$

$$s(k) = -\frac{1}{K_d} \left\{ \ln \left[e^{-K_d s(k-1)} -(1-e^{-K_d t})/N\right] \right\} , \quad k > 1.$$

The integrals are evaluated by means of the Trapezoidal Rule.

The integrals were evaluated for N varying between 50 and 300 in increments of 10, by each of the four methods. Using values obtained from Equations (4.101) and (4.103) (8,000 steps) as correct solution, per cent errors were calculated for each method. The execution time for a fixed number of intervals varies from method to method. Thus, comparison of error as a function of number of intervals is inadequate as a means of comparing the various methods. Instead the time required for each method for each N was measured and the methods were compared as a function of execution time.

Table 1 gives the per cent error for each method for varying execution times. Time is measured in 1/60 of a second. Since it was felt that it would be impractical to perform the BOD and OD integrations by different methods, the times listed in the table is the total required to complete both integrations. The per cent error is the ratio of the total error to the sum of the correct values.

As can be seen from the table, method 4 is uniformly superior to the other methods tested. In addition method 4 offers two advantages which are not considered in the above comparison. All of the previous work has been in the transient state; a steady state was approached by choosing the time lapse to be sufficiently large. Method 4 can be modified to perform the necessary integration from zero to infinity by choosing the points for evaluation as follows:

$$s(1) = -\frac{1}{K_d} \ln \left(\frac{N-1}{N}\right) ,$$

$$s(k) = -\frac{1}{K_d} \ln \left(e^{-K_d s(k-1)} - 1/N\right), \quad k > 1 .$$

Thus, the steady state solutions can be obtained directly.

The times listed in Table 1 are the times required to locate the points s(1)...s(N) as well as to perform the integrations. For this reason the time required to compute the integrals by method 4 for a fixed N was about 20% greater than the time required by the Trapezoidal Rule for the same N. The general situation is to evaluate the integrals for incrementing values of distance or time, with all other parameters remaining constant. The points s(1)...s(N) need only to be evaluated once for each set of parameters. This will cut the time required for integration by about 20%.

TABLE 1

COMPARISON OF DIFFERENT METHODS OF INTEGRATION

Time (1/60 Sec.)	Percent Error for Simpson's Rule	Percent Error for Trapezodial Rule	Percent Error for Hermitian Formula	Percent Error for Trap. Rule with Inc. Inc.
40	9.92	10.25	17.65	1.30
50	4.17	7.71	14.19	0.80
60	1.82	5.61	10.67	0.45
70	1.20	4.45	8.36	0.27
80	0.87	3.27	6.93	0.19
90	0.38	2.15	5.57	0.20
100	0.67	1.38	4.58	0.13
110	1.08	0.78	3.78	0.15
120	0.34	0.36	3.07	0.10
130	1.57	0.08	2.39	0.09
140	1.72	0.17	1.89	0.09
150	1.72	0.32	1.46	0.06

LAND RUNOFF AND BENTHAL DEMAND

Difference Equations

In the preceding analysis the increase of BOD by land runoff, and the decrease of DO by the benthal demand were ignored. The effects of these factors on the BOD and DO profile may be calculated separately, and the result superimposed on the over-all profile. Since the land runoff and the benthal demand are assumed to be uniform along the stretch of the estuary, it is not necessary to consider the time variable velocity. The BOD and DO profiles produced by these factors will be uniform along the stretch at any time t, where t is the time lapse since the runoff began, i.e., land runoff and benthal demand in an estuary is analogous to a stream. The stochastic approach to these factors will be similar to the approach of Thayer and Krutchkoff.

Following Thayer and Krutchkoff, we assume that:

1. Pr[L increased by an amount Δ due to L_a in time δt] = $L_a \delta t/\Delta$, where L is the BOD concentration at the time of transition, and L_a is the rate of land runoff.

2. Pr[oxygen deficiency increases by an amount Δ due to D_B in time δt] = $D_B \delta t/\Delta$, where D_B is the decrease in DO due to the benthal demand.

Both of these factors are assumed to be independent of the level of BOD and DO at the time of transition.

Again following Thayer and Krutchkoff, the probability of BOD and DO decreasing due to the bacterial action is assumed to be proportional to the BOD level at the time of transition. Also, the probability of DO increasing due to reaeration is directly proportional to the oxygen deficiency. (These statements can be derived from the earlier assumptions concerning the decay probabilities as given in Diffusion Model, in the section on Difference Equations).

Let $p_n(t)$ = Pr[the system is in BOD state n at time t]. Recall that the only source of BOD considered is the land runoff. The system can be in state n at time t $+\delta t$ in only three ways:

1. The system is in BOD state n-1 at time t and BOD is increased by Δ in time δt due to L_a.

2. The system is in BOD state n+1 at time t and the BOD is decreased by an amount Δ in time δt due to the bacterial action or sedimentation.

3. The system is in BOD state n at time t and there is no change.

All other possibilities are of order δt^2, which is considered negligible. Obviously, if $n = 0$ case 1 is eliminated.

Therefore,

$$p_n(t + \delta t) = p_{n-1}(t) \frac{L_a}{\Delta} \delta t + p_{n+1}(t) (n+1) K_d \delta t$$

$$+ p_n(t) \left[1 - \frac{L_a}{\Delta} \delta t - nK_d \delta t\right] \text{ for } n \geqslant 1, \tag{6.1}$$

and

$$p_o(t+\delta t) = p_1 K_d \delta t + p_o(t) \left[1 - \frac{L_a}{\Delta} \delta t \right] . \tag{6.2}$$

Expanding in a Taylor series and taking the limit as $\delta t \to 0$, Equations (6.1) and (6.2) become,

$$\frac{dp_n(t)}{dt} + p_n(t) \left(\frac{L_a}{\Delta} + n K_d\right) = p_{n-1} \frac{L_a}{\Delta} + p_{n+1}(t) (n+1) K_d$$

for

$$n \geqslant 1 \tag{6.3}$$

and

$$\frac{dp_o(t)}{dt} + \frac{L_a}{\Delta} p_o(t) = p_1(t) K_d \quad . \qquad (6.4)$$

Before solving these differential equations, similar equations for the oxygen deficiency will be developed.

Let $q_n(t)$ = Pr[the system is in the OD state n at time t]. The ways in which the system can be in state n at time $t + \delta t$ are as follows:

1. The system is in OD state n-1 at time t and the OD is increased by Δ in time δt by D_B.

2. The system is in OD state n-1 at time t and the OD is increased by Δ in time δt due to the bacterial decay of a BOD particle. The probability of a BOD particle being acted upon by the bacteria given that the BOD is in state k is $K_1 k p_k(t)\, \delta t$. Therefore, the probability of an increase of one OD particle due to bacterial action is $\sum_{k=0}^{\infty} k p_k(t) K_1 \delta t$. Note if E(n) is the expected BOD state the above probability becomes $E(n) K_1 \, \delta t$.

3. The system is at OD state n+1 at t and the OD is decreased by Δ due to reaeration.

4. The system is in OD state n at t and no change occurs.

Obvious modifications are again necessary for n = 0. Then

$$q_n(t + \delta t) = q_{n-1}(t) \left[\frac{D_B}{\Delta} \delta t + E(n) K_1 \delta t \right]$$

$$+ q_{n+1}(t) (n+1) K_2 \delta t$$

$$+ \; q_n(t) \; [\, 1 - \frac{D_B}{\Delta} \, \delta t - n\, K_2 \, \delta t - E(n)\, K_1 \, \delta t \,] \; ,$$

for

$$n \geqslant 1 \qquad (6.5)$$

and

$$q_o(t + \delta t) = q_1(t) \; K_2 \, \delta t$$

$$+ \; q_o(t) \; [\, 1 - \frac{D_B}{\Delta} \delta t - E(n) \; K_1 \, \delta t \,] \, . \qquad (6.6)$$

As before,

$$\frac{d\, q_n\,(t)}{dt} + q_n(t) \; [\, \frac{D_B}{\Delta} + n\, K_2 + E(n)\, K_1 \,]$$

$$= q_{n-1}(t) \; [\, \frac{D_B}{\Delta} + E(n) \; K_1 \,] + q_{n+1}(t) \; (n+1) K_2$$

for

$$n \geqslant 1 \tag{6.7}$$

and

$$\frac{d\, q_o(t)}{dt} + q_o(t) \left[\frac{D_B}{\Delta} + E(n)\; K_1 \right] = q_1(t)\, K_2 \;. \tag{6.8}$$

Solution to Differential Equations

Suppose one has the following set of differential equations:

$$\frac{d\, p_o(t)}{dt} + \lambda\, p_o(t) = \mu\, p_1\,(t) \;, \tag{6.9}$$

and

$$\frac{d\, p_n(t)}{dt} + (\lambda + n\mu)\, p_n(t) = \lambda\, P_{n-1}(t) + (n+1)\mu p_{n+1}(t)$$

for

$$n = 1, 2, 3, \ldots, \tag{6.10}$$

with initial conditions

$$p_m(0) = 1 \ , \tag{6.11}$$

$$p_n(0) = 0 \text{ for } n \neq m \ .$$

Let

$$P(x,t) = \sum_{n=0}^{\infty} x^n \, p_n(t) \tag{6.12}$$

be the generating function of $p_n(t)$. From Equations (6.9) through (6.12), one obtains

$$\frac{\partial P(x,t)}{\partial t} + \mu(x-1)\frac{\partial P(x,t)}{\partial x} = \lambda(x-1)\,P(x,t) \ . \tag{6.13}$$

Consider the auxiliary equation

$$\frac{dt}{1} = \frac{dx}{\mu(x-1)} = \frac{dP(x,t)}{\lambda(x-1)\,P(x,t)} \tag{6.14}$$

It is a general theorem of differential equations that $F(u,v) = 0$ is a solution to Equation (6.13) where F is an arbitrary function, and $u(x,t,P) = c_1$, and $v(x,t,P) = c_2$ are particular solutions to Equation (6.14) (54).

First, consider the equation

$$\frac{dx}{\mu\,(x-1)} = \frac{dP}{\lambda\,(x-1)\,P} \qquad (6.15)$$

Here,

$$c_1 = P\,(x,t)\,e^{-px} ,$$

where $P \doteq \lambda/\mu$ is a particular solution to Equation (6.15).

Secondly, consider

$$dt = \frac{dx}{\mu\,(x-1)} , \qquad (6.16)$$

which has $c_2 = e^{\mu t}/(1-x)$ as a particular solution.

Therefore, the general solution to Equation (6.13) is

$$F\,[P\,e^{-\rho x} , e^{\mu t}/(1-x)\,] = 0 . \qquad (6.17)$$

Equation (6.17) is equivalent to

$$P\,(x,t)\;e^{-\rho x} = \psi\;[\,e^{\mu t}/(1-x)\,] , \qquad (6.18)$$

where Ψ is an arbitrary function.

Now initial condition (6.11) gives

$$P(x,0) = x^{m} \quad , \tag{6.19}$$

and therefore

$$x^{m} e^{-\rho x} = \psi\,[1/(1-x)] \quad . \tag{6.20}$$

Letting $y = 1/(1-x)$ yields

$$(1-1/y)^{m} e^{-\rho(1-1/y)} = \psi(y) \quad , \tag{6.21}$$

and hence,

$$\begin{aligned} P(x,t) &= e^{\rho x}\,[1-(1-x)e^{-\mu t}]^{m}\, e^{-\rho(1-(1-x)e^{-\mu t})} \\ &= (1-e^{-\mu t})^{m}\,\left[1+\frac{x e^{-\mu t}}{1-e^{-\mu t}}\right]^{m} e^{-\rho(1-e^{-\mu t})}\, e^{\rho(1-e^{-\mu t})x}. \end{aligned} \tag{6.22}$$

In order to simplify Equation (6.22), let $\zeta(t) = 1\text{-}e^{-\mu t}$. Expanding the binomial and exponential terms in x yields

$$P(x,t) = e^{-\rho\zeta}\,\zeta^m \left[\sum_{i=0}^{m}\binom{M}{i}\frac{x^i e^{-\mu t i}}{\zeta^i}\right]\left[\sum_{j=0}^{\infty}\frac{(\rho\zeta x)^j}{j!}\right]. \qquad (6.23)$$

Extracting the coefficient of x^n, one obtains

$$p_n(t) = \zeta^m\, e^{-\rho\zeta}\sum_{k=0}^{M}\binom{m}{k}\frac{(\rho\zeta)^{n-k}}{\zeta^k (n-k)!}\, e^{-\mu tk}, \qquad (6.24)$$

where M = max(m,n).

Application of The Solution to Differential Equations to BOD at OD Equations

Equations (6.3) and (6.4) are in the same form as Equations (6.9) and (6.10), where $\lambda = \frac{L_a}{\Delta}$ and $\mu = K_d$. Let $\rho = L_a/K_d\Delta$. Therefore, the solution to the set of Equations (6.3) and (6.4) is

$$p_n(t) = \zeta^m\, e^{-\rho\zeta}\sum_{k=0}^{M}\binom{m}{k}\frac{(\rho\zeta)^{n-k}}{\zeta^k (n-k)!}\, e^{-K_d tk}, \qquad (6.25)$$

where m is the BOD state at t = 0.

E(n), the expected BOD state, can be found by taking the derivative of P(x,t) with respect to x; and evaluating at x = 1. This yields

$$\frac{\partial P(x,t)}{\partial x} = \zeta e^{-\rho(1-x)} (\zeta + xe^{-\mu t})^{m-1}$$

$$[\rho\zeta + e^{-\mu t}(\rho x + m/\zeta)] , \qquad (6.26)$$

and

$$E(n) = \frac{\partial P(x,t)}{\partial x}\Big|_{x=1} = \zeta(\zeta + e^{-\mu t})^{m-1} [\rho\zeta + e^{-\mu t}(\rho + m/\zeta)] . \qquad (6.27)$$

Similarly, if in Equations (6.7) and (6.8) $\lambda = \frac{D_B}{\Delta} + E(n) K_1$, where E(n) is defined in (6.27), $\mu = K_2$, and $\rho = (\frac{D_B}{\Delta} + E(n) K_1)/K_2$, then

$$q_n(t) = \zeta^{\underset{\sim}{m}} e^{-\underset{\sim}{\rho}\zeta} \sum_{k=0}^{M} \binom{\underset{\sim}{m}}{k} \frac{(\rho\zeta)^{n-k}}{\zeta^k (n-k)!} e^{-K_2 tk} , \qquad (6.28)$$

where $\underline{m}$ is the OD state at t = 0

In our application steady state conditions may be assumed. It is quite reasonable to assume that the land runoff and the benthal demand have existed for a sufficient length of time for steady state conditions to arise; even if the major sources are transient in nature. This simplifies Equations (6.25) and (6.28), since as $t \to \infty$,

$$\zeta = 1 - e^{-\mu t} \to 1 .$$

So, if one lets

$$\lim_{t \to \infty} p_n(t) = \bar{p}_n \ , \text{and} \ \lim_{t \to \infty} q_n(t) = \bar{q}_n \ ,$$

one obtains

$$\bar{p}_n = \frac{\rho^n e^{-\rho}}{n!} \ , \tag{6.29}$$

and

$$\bar{q}_n = \frac{\underline{\rho}^n \ e^{-\underline{\rho}}}{n!} \ . \tag{6.30}$$

Thus, BOD and OD are distributed as Poisson distributions with parameters

$$\rho = L_a/K_d \Delta \quad ,$$

and

$$\underline{\rho} = \frac{K_d D_B + K_1 L_a}{\Delta K_2 K_d} \quad ,$$

respectively.

The analysis of the Land Runoff and Benthal Demand, shows that the BOD and OD distribution due to all factors except L_a and D_B were Poisson distributions with parameters m_1 and m_2, respectively. Therefore, the convolutions are Poisson distributions with parameters $m_1 + \rho$ and $m_2 + \underline{\rho}$, respectively. These are the distributions for the BOD and OD states. The concentrations are obtained by multiplying the states by Δ.

EVALUATION OF MODELS

Numerical Results for Diffusion Model

Two distinct stochastic models for determining the BOD and DO probability distributions have been developed. At this time it is appropriate to make some statements concerning the degree of compatibility of the models. Several valuable conclusions can be obtained by comparing numerical results. Figures 45 through 68 show the mean BOD and OD temporal and spatial profiles obtained from the Diffusion Model for the same hypothetical situation as in the Non-Dispersion Model in the section Numerical Results for the Non-Dispersion Model. The diffusion coefficient for Figures 45 through 52 is 0.5 mi.^{2}/day, for Figures 53 through 60, E is 2 mi.^{2}/day, and E is taken as 5 mi.^{2}/day in Figures 61 through 68. The programs used to determine the profile appear in Appendix C.

These curves represent the predicted average BOD and OD values as functions of distance or time. The professed merit of the models developed in this paper is the ability to predict the entire probability distribution. Following the analysis of the preceding chapters, the probability distribution for any point in time and space can be obtained from a table of Poisson distributions. However, in order to have a quantitative picture of the entire estuary environment it is necessary to consider many points. Since the mean BOD and OD concentrations are functions of time and space, the probability distributions are also temporally and spacially dependent. Collecting the distribution information for each point, and reaching workable conclusions would be a task. In order to utilize the results as a tool in the overall decision process it is necessary to summarize in a concise form.

The procedure adopted is illustrated in Tables 2 and 3, which is a print out from Program 11, Appendix C. This summarizes the information for a given point in space, in this case 5 miles downstream from the source. The mean BOD and OD levels are obtained for each increment of the tidal cycle. In addition, the probability of the concentration being above several critical levels at each instant in time is given. Finally, and most importantly, the expected percentage of time that the concentration is above each critical value is obtained. Thus, even though the program generates vast quantities of information internally, an easily assimilated summation is available to the decision maker. For design purposes the program may be executed repeatedly, varying the pertinent parameters, and the expected increase or decrease in water quality may be easily noted.

For the illustrative example shown in Tables 2 and 3, the estuary parameters are the same as in the earlier examples. The table represents a point 5 miles downstream from the source and Δ is 0.1. The increment in time is 0.005 days.

Comparison of Models

In an engineering problem such as this the ultimate criterion as to the degree of success or failure of a mathematical model is its agreement with the real world. A verification of this type is given in the section, Verification of Diffusion Model. However, several conclusions may be obtained by considering the mathematical analysis, and the hypothetical data for the two models.

It will be recalled that the assumptions required in the diffusion model were less constraining than in the non-dispersion model. Consequently, the analysis involved was more complex. However, this is of no concern to the field engineer utilizing the results.

A quick comparison of the numerical results reveals that the traumatic peaks observed for the non-dispersion model are obliterated in the diffusion model, even where the diffusion coefficient is as small as 0.5 mi.^{2}/day. Earlier, stream examples were cited in which a diffusion coefficient as large as 1.5 mi.^{2}/day could safely be neglected. This appears not to be the case in an estuary.

To understand the reason for this, it is necessary to consider the physical differences between a stream and an estuary. For the moment, think of the uniform source as emitting discrete but closely spaced slugs of pollution. In a stream with a constant flow, the concentration resulting from each slug is the same. The resulting concentration is given by Equation (3.3). If a slug diffuses, the peak concentration will decrease and pollution will spill into adjacent segments. However, with a continuous source the adjacent segment will also contain slugs of pollution which in turn diffuses into the original segment. In short, the concentration loss due to diffusion from a segment is approximately equal to the gain from neighboring segments. The total effect is not appreciably deviated from the non-diffusion situation.

In contrast to a stream, the varying estuary velocity produces varying concentration levels. Thus, the diffusion loss suffered by a segment with a high concentration is not offset by gains from its lower concentration neighbors. An inspection of the curves in the Non-Dispersion Model section reveals that even though the peaks are dramatic, they are extremely narrow in the longitudinal direction. Thus, it is not difficult to imagine their rapid dissipation even when diffusion is not pronounced.

In light of the numerical results and the above discussion, one must conclude that the non-dispersion model is inadequate to describe the estuary situation. Fortunately, this is not a serious loss since the model possesses no significant advantages over the diffusion model. It should be noted that the non-dispersion model with the line source approach yields results which are compatible with the diffusion model. Therefore, this approach can not be summarily dismissed on the same grounds as the earlier approaches. Again, there are no apparent advantages to the line source approach. Diffusion is introduced in an artificial manner and the assumptions are just as stringent as those of the diffusion model. A possible advantage of the line source approach will be discussed in the next section.

Segmenting an Estuary

The constraints placed upon the physical system outlined in the section Scope and Limitations of the Introduction, require that the physical parameters remain invariant in time and space. Temporal invariance is reasonable (except for photosynthetic effects which are discussed in the section, Verification of Diffusion Model), but spatial invariance is rarely realized. Width and depth vary markedly between the foot and the mouth of a large estuary, and several of the parameters are functions of the geometry of the estuary.

Spatial invariance in the parameters is generally required in the development of the deterministic models. However, the problem is circumvented by discretizing the given estuary into segments. The equations are applied to each segment individually, assuming invariance within each segment while varying the parameters for different segments. This approach is illustrated in Thomann's work which was discussed in the section Thomann's Model in the Literature Review.

The stochastic models as developed in this paper cannot be directly applied to the segmented situation. Work on possible appropriate modifications is pending. For this reason, it is felt that the non-dispersion model with the line source approach should not be categorically eliminated. At this point, it is not apparent which model will most readily lend itself to the segmented situation.

TABLE 2

SUMMARY OF BOD PROBABILITY DISTRIBUTION
FIVE MILES DOWNSTREAM FROM THE SOURCE

TIME IN HRS.	MEAN BOD LEVEL	PROBABILITY BOD IS ABOVE 2.50	3.00	3.50	4.00
0.0	2.09	0.16	0.02	0.00	0.0
0.12	2.16	0.20	0.03	0.00	0.00
0.24	2.22	0.24	0.05	0.00	0.00
0.36	2.29	0.29	0.06	0.01	0.00
0.48	2.36	0.34	0.08	0.01	0.00
0.60	2.44	0.40	0.11	0.02	0.00
0.72	2.51	0.45	0.14	0.02	0.00
0.84	2.58	0.51	0.18	0.03	0.00
0.96	2.66	0.57	0.22	0.05	0.01
1.08	2.74	0.63	0.27	0.06	0.01
1.20	2.81	0.68	0.32	0.09	0.01
1.32	2.89	0.73	0.38	0.11	0.02
1.44	2.97	0.78	0.43	0.15	0.03
1.56	3.05	0.82	0.49	0.18	0.04
1.68	3.13	0.85	0.55	0.22	0.05
1.80	3.20	0.88	0.60	0.26	0.07
1.92	3.27	0.90	0.64	0.31	0.09
2.04	3.34	0.92	0.68	0.35	0.11
2.16	3.40	0.93	0.72	0.38	0.13
2.28	3.45	0.94	0.74	0.42	0.15
2.40	3.49	0.95	0.77	0.45	0.17
2.52	3.52	0.95	0.78	0.47	0.19
2.64	3.55	0.96	0.80	0.49	0.20
2.76	3.57	0.96	0.81	0.50	0.21
2.88	3.58	0.96	0.81	0.51	0.21
3.00	3.58	0.96	0.81	0.51	0.21
3.12	3.57	0.96	0.81	0.50	0.21
3.24	3.56	0.96	0.80	0.50	0.20
3.36	3.54	0.96	0.79	0.48	0.19
3.48	3.51	0.95	0.78	0.46	0.18
3.60	3.48	0.95	0.76	0.44	0.17
3.72	3.44	0.94	0.74	0.41	0.15
3.84	3.39	0.93	0.71	0.38	0.13
3.96	3.34	0.92	0.68	0.35	0.11
4.08	3.28	0.90	0.65	0.31	0.09
4.20	3.22	0.88	0.61	0.27	0.08
4.32	3.15	0.86	0.56	0.23	0.06

TABLE 2B

SUMMARY OF BOD PROBABILITY DISTRIBUTION

FIVE MILES DOWNSTREAM FROM THE SOURCE

TIME IN HRS.	MEAN BOD LEVEL	PROBABILITY BOD IS ABOVE 2.50	3.00	3.50	4.00
4.44	3.08	0.83	0.51	0.20	0.05
4.56	3.01	0.80	0.46	0.16	0.03
4.68	2.93	0.76	0.40	0.13	0.02
4.80	2.86	0.71	0.35	0.10	0.02
4.92	2.78	0.66	0.29	0.08	0.01
5.04	2.70	0.60	0.24	0.06	0.01
5.16	2.62	0.54	0.20	0.04	0.00
5.28	2.54	0.48	0.16	0.03	0.00
5.40	2.46	0.42	0.12	0.02	0.00
5.52	2.39	0.36	0.09	0.01	0.00
5.64	2.31	0.30	0.07	0.01	0.00
5.76	2.24	0.25	0.05	0.00	0.00
5.88	2.17	0.20	0.03	0.00	0.00
6.00	2.10	0.16	0.02	0.00	0.0
6.12	2.03	0.13	0.02	0.00	0.0
6.24	1.97	0.10	0.01	0.00	0.0
6.36	1.91	0.07	0.01	0.00	0.0
6.48	1.85	0.06	0.00	0.00	0.0
6.60	1.79	0.04	0.00	0.00	0.0
6.72	1.74	0.03	0.00	0.0	0.0
6.84	1.69	0.02	0.00	0.0	0.0
6.96	1.64	0.02	0.00	0.0	0.0
7.08	1.60	0.01	0.00	0.0	0.0
7.20	1.55	0.01	0.00	0.0	0.0
7.32	1.51	0.01	0.00	0.0	0.0
7.44	1.48	0.01	0.00	0.0	0.0
7.56	1.44	0.00	0.00	0.0	0.0
7.68	1.41	0.00	0.00	0.0	0.0
7.80	1.39	0.00	0.00	0.0	0.0
7.92	1.36	0.00	0.00	0.0	0.0
8.04	1.34	0.00	0.00	0.0	0.0
8.16	1.32	0.00	-0.00	0.0	0.0
8.28	1.30	0.00	-0.00	0.0	0.0
8.40	1.29	0.00	-0.00	0.0	0.0
8.52	1.27	0.00	-0.00	0.0	0.0
8.64	1.26	0.00	-0.00	0.0	0.0
8.76	1.26	0.00	-0.00	0.0	0.0

TABLE 2C

SUMMARY OF BOD PROBABILITY DISTRIBUTION

FIVE MILES DOWNSTREAM FROM THE SOURCE

TIME IN HRS.	MEAN BOD LEVEL	PROBABILITY BOD IS ABOVE 2.50	3.00	3.50	4.00
8.88	1.25	0.00	-0.00	0.0	0.0
9.00	1.25	0.00	-0.00	0.0	0.0
9.12	1.25	0.00	-0.00	0.0	0.0
9.24	1.25	0.00	-0.00	0.0	0.0
9.36	1.26	0.00	-0.00	0.0	0.0
9.48	1.27	0.00	-0.00	0.0	0.0
9.60	1.28	0.00	-0.00	0.0	0.0
9.72	1.29	0.00	-0.00	0.0	0.0
9.84	1.31	0.00	-0.00	0.0	0.0
9.96	1.33	0.00	-0.00	0.0	0.0
10.08	1.35	0.00	0.00	0.0	0.0
10.20	1.38	0.00	0.00	0.0	0.0
10.32	1.40	0.00	0.00	0.0	0.0
10.44	1.43	0.00	0.00	0.0	0.0
10.56	1.47	0.00	0.00	0.0	0.0
10.68	1.50	0.01	0.00	0.0	0.0
10.80	1.54	0.01	0.00	0.0	0.0
10.92	1.58	0.01	0.00	0.0	0.0
11.04	1.63	0.02	0.00	0.0	0.0
11.16	1.68	0.02	0.00	0.0	0.0
11.28	1.73	0.03	0.00	0.0	0.0
11.40	1.78	0.04	0.00	0.00	0.0
11.52	1.84	0.05	0.00	0.00	0.0
11.64	1.90	0.07	0.01	0.00	0.0
11.76	1.96	0.10	0.01	0.00	0.0
11.88	2.02	0.12	0.02	0.00	0.0
12.00	2.09	0.16	0.02	0.00	0.0

THE EXPECTED PERCENT OF TIME THAT THE BOD LEVEL WILL BE ABOVE 2.50 MG./L. IS 36.05 %.

THE EXPECTED PERCENT OF TIME THAT THE BOD LEVEL WILL BE ABOVE 3.00 MG./L. IS 22.26 %.

THE EXPECTED PERCENT OF TIME THAT THE BOD LEVEL WILL BE ABOVE 3.50 MG./L. IS 10.68 %.

THE EXPECTED PERCENT OF TIME THAT THE BOD LEVEL WILL BE ABOVE 4.00 MG./L. IS 3.60 %.

TABLE 3

SUMMARY OF OD PROBABILITY DISTRIBUTION

FIVE MILES DOWNSTREAM FROM THE SOURCE

TIME IN HRS.	MEAN OD LEVEL	PROBABILITY OD IS ABOVE			
		2.00	2.50	3.00	3.50
0.0	2.20	0.61	0.22	0.04	0.00
0.12	2.23	0.64	0.25	0.05	0.00
0.24	2.27	0.67	0.27	0.06	0.01
0.36	2.30	0.69	0.29	0.06	0.01
0.48	2.33	0.71	0.32	0.07	0.01
0.60	2.36	0.73	0.34	0.08	0.01
0.72	2.39	0.75	0.36	0.09	0.01
0.84	2.42	0.77	0.38	0.10	0.01
0.96	2.44	0.78	0.40	0.11	0.02
1.08	2.47	0.80	0.42	0.12	0.02
1.20	2.49	0.81	0.44	0.13	0.02
1.32	2.51	0.82	0.46	0.14	0.02
1.44	2.53	0.83	0.47	0.15	0.03
1.56	2.55	0.84	0.49	0.16	0.03
1.68	2.57	0.85	0.50	0.17	0.03
1.80	2.58	0.85	0.51	0.18	0.03
1.92	2.59	0.86	0.52	0.18	0.04
2.04	2.61	0.86	0.53	0.19	0.04
2.16	2.62	0.87	0.54	0.20	0.04
2.28	2.63	0.87	0.55	0.20	0.04
2.40	2.63	0.88	0.55	0.21	0.04
2.52	2.64	0.88	0.56	0.21	0.04
2.64	2.65	0.88	0.56	0.21	0.04
2.76	2.65	0.88	0.56	0.21	0.05
2.88	2.65	0.88	0.57	0.22	0.05
3.00	2.65	0.88	0.57	0.22	0.05
3.12	2.65	0.88	0.57	0.22	0.05
3.24	2.65	0.88	0.57	0.22	0.05
3.36	2.65	0.88	0.56	0.21	0.05
3.48	2.65	0.88	0.56	0.21	0.04
3.60	2.64	0.88	0.56	0.21	0.04
3.72	2.63	0.88	0.55	0.21	0.04
3.84	2.63	0.87	0.55	0.20	0.04
3.96	2.62	0.87	0.54	0.20	0.04
4.08	2.61	0.86	0.53	0.19	0.04
4.20	2.59	0.86	0.52	0.18	0.04
4.32	2.58	0.85	0.51	0.18	0.03

TABLE 3B

SUMMARY OF OD PROBABILITY DISTRIBUTION

FIVE MILES DOWNSTREAM FROM THE SOURCE

TIME IN HRS.	MEAN OD LEVEL	PROBABILITY OD IS ABOVE			
		2.00	2.50	3.00	3.50
4.44	2.56	0.85	0.50	0.17	0.03
4.56	2.54	0.84	0.48	0.16	0.03
4.68	2.53	0.83	0.47	0.15	0.03
4.80	2.50	0.82	0.45	0.14	0.02
4.92	2.48	0.80	0.43	0.13	0.02
5.04	2.46	0.79	0.41	0.12	0.02
5.16	2.43	0.78	0.39	0.11	0.02
5.28	2.40	0.76	0.37	0.10	0.01
5.40	2.37	0.74	0.35	0.09	0.01
5.52	2.35	0.72	0.33	0.08	0.01
5.64	2.31	0.70	0.30	0.07	0.01
5.76	2.28	0.68	0.28	0.06	0.01
5.88	2.25	0.65	0.26	0.05	0.01
6.00	2.22	0.63	0.23	0.04	0.00
6.12	2.19	0.60	0.21	0.04	0.00
6.24	2.15	0.57	0.19	0.03	0.00
6.36	2.12	0.55	0.17	0.03	0.00
6.48	2.09	0.52	0.15	0.02	0.00
6.60	2.05	0.49	0.14	0.02	0.00
6.72	2.02	0.46	0.12	0.02	0.00
6.84	1.99	0.43	0.11	0.01	0.00
6.96	1.96	0.40	0.09	0.01	0.00
7.08	1.93	0.38	0.08	0.01	0.00
7.20	1.90	0.35	0.07	0.01	0.00
7.32	1.87	0.33	0.06	0.01	0.00
7.44	1.84	0.31	0.06	0.00	0.00
7.56	1.82	0.28	0.05	0.00	0.00
7.68	1.80	0.27	0.04	0.00	0.00
7.80	1.77	0.25	0.04	0.00	0.0
7.92	1.75	0.23	0.03	0.00	0.0
8.04	1.73	0.22	0.03	0.00	0.0
8.16	1.72	0.21	0.03	0.00	0.0
8.28	1.70	0.20	0.03	0.00	0.0
8.40	1.69	0.19	0.02	0.00	0.0
8.52	1.68	0.18	0.02	0.00	0.0
8.64	1.67	0.18	0.02	0.00	0.0
8.76	1.67	0.17	0.02	0.00	0.0

TABLE 3C

SUMMARY OF OD PROBABILITY DISTRIBUTION

FIVE MILES DOWNSTREAM FROM THE SOURCE

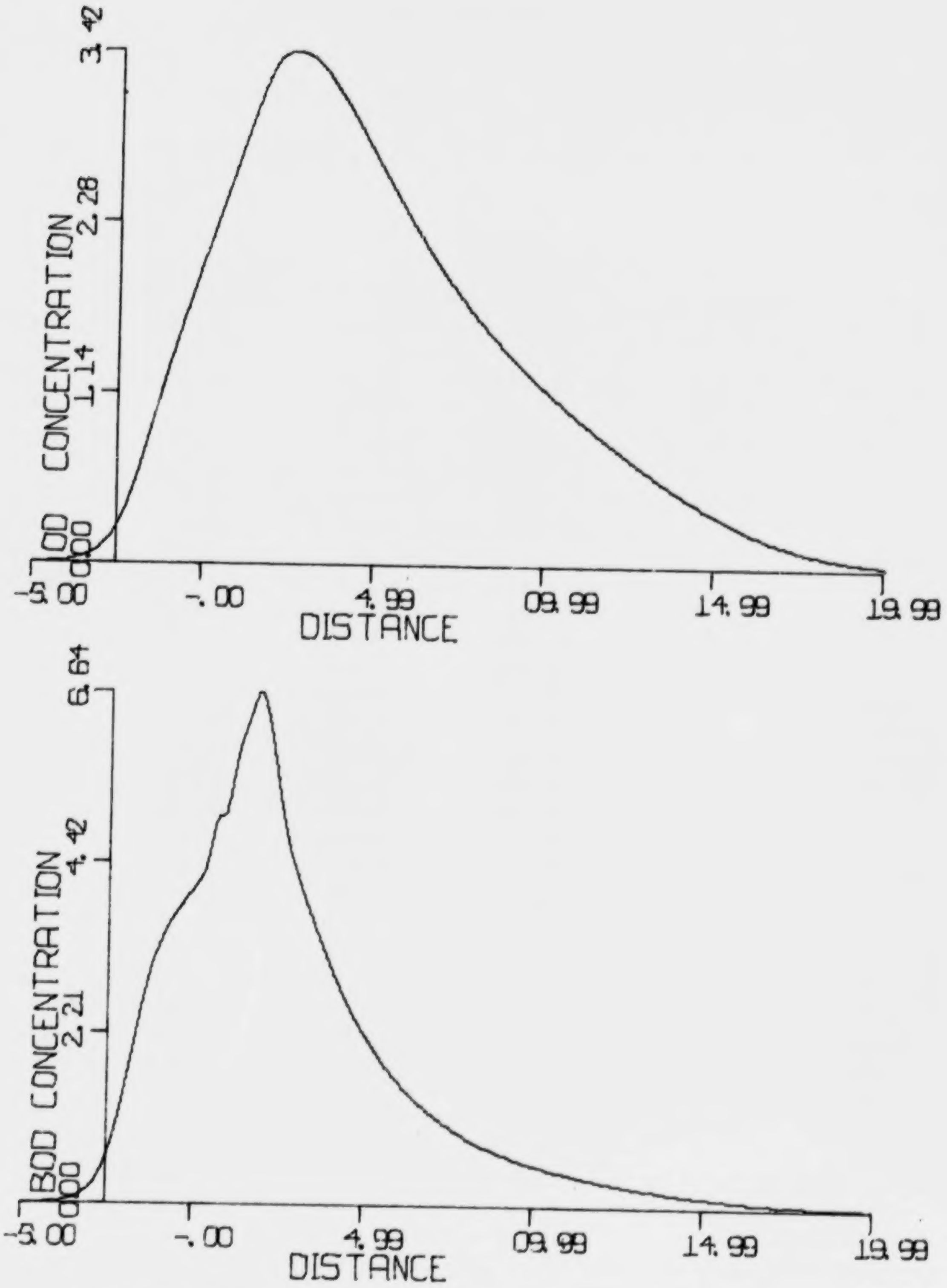
TIME IN HRS.	MEAN OD LEVEL	PROBABILITY OD IS ABOVE 2.00	2.50	3.00	3.50
8.88	1.66	0.17	0.02	0.00	0.0
9.00	1.66	0.17	0.02	0.00	0.0
9.12	1.66	0.17	0.02	0.00	0.0
9.24	1.66	0.17	0.02	0.00	0.0
9.36	1.67	0.17	0.02	0.00	0.0
9.48	1.67	0.18	0.02	0.00	0.0
9.60	1.68	0.18	0.02	0.00	0.0
9.72	1.69	0.19	0.02	0.00	0.0
9.84	1.70	0.20	0.03	0.00	0.0
9.96	1.72	0.21	0.03	0.00	0.0
10.08	1.73	0.22	0.03	0.00	0.0
10.20	1.75	0.23	0.03	0.00	0.0
10.32	1.77	0.25	0.04	0.00	0.0
10.44	1.80	0.27	0.04	0.00	0.00
10.56	1.82	0.28	0.05	0.00	0.00
10.68	1.84	0.31	0.06	0.00	0.00
10.80	1.87	0.33	0.06	0.01	0.00
10.92	1.90	0.35	0.07	0.01	0.00
11.04	1.93	0.38	0.08	0.01	0.00
11.16	1.96	0.41	0.10	0.01	0.00
11.28	1.99	0.44	0.11	0.01	0.00
11.40	2.03	0.47	0.13	0.02	0.00
11.52	2.06	0.50	0.14	0.02	0.00
11.64	2.10	0.53	0.16	0.02	0.00
11.76	2.13	0.56	0.18	0.03	0.00
11.88	2.17	0.58	0.20	0.03	0.00
12.00	2.20	0.61	0.22	0.04	0.00

THE EXPECTED PERCENT OF TIME THAT THE OD LEVEL
WILL BE ABOVE 2.00 MG./L. IS 56.74 %.

THE EXPECTED PERCENT OF TIME THAT THE OD LEVEL
WILL BE ABOVE 2.50 MG./L. IS 26.37 %.

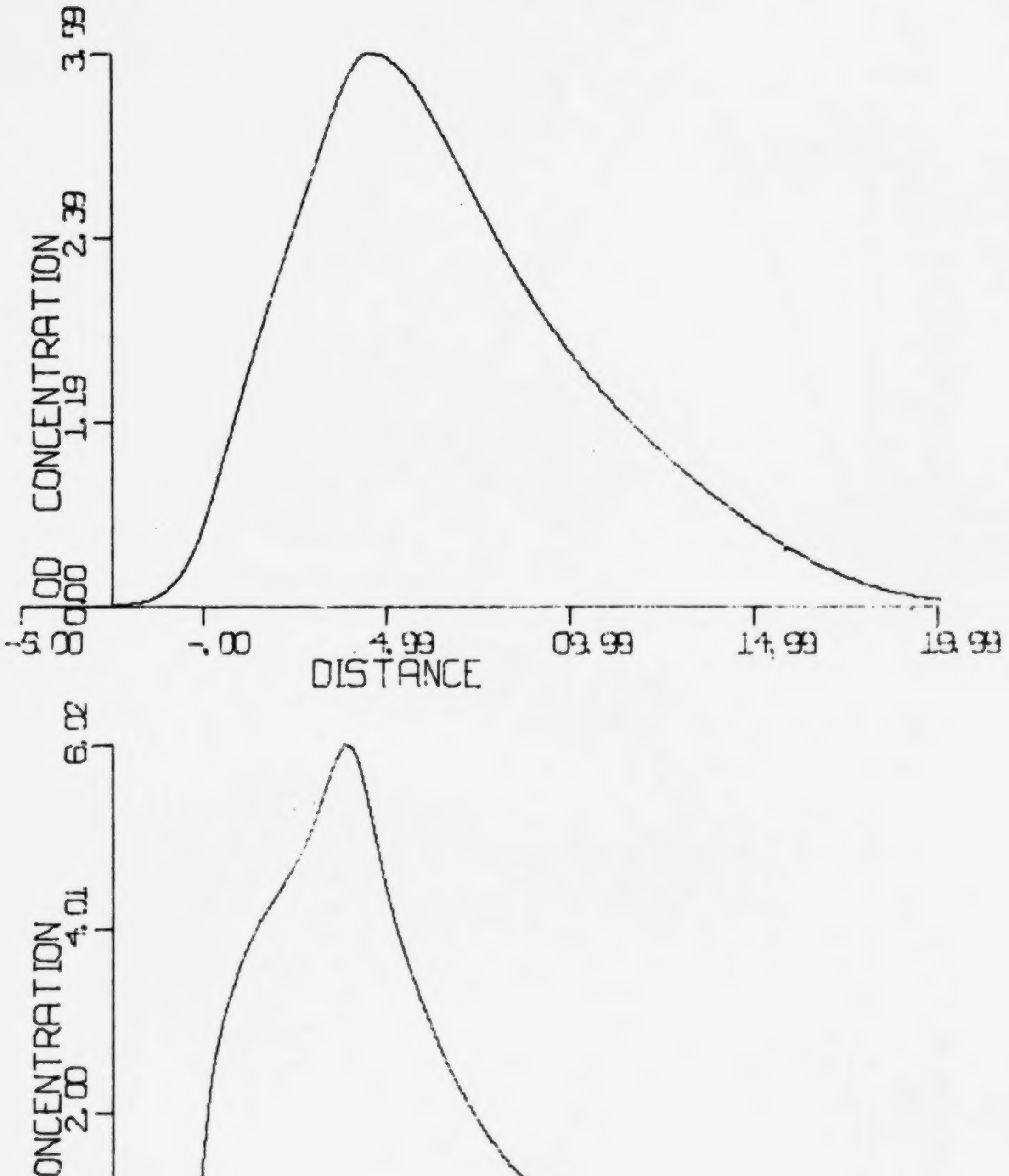
THE EXPECTED PERCENT OF TIME THAT THE OD LEVEL
WILL BE ABOVE 3.00 MG./L. IS 7.82 %.

THE EXPECTED PERCENT OF TIME THAT THE OD LEVEL
WILL BE ABOVE 3.50 MG./L. IS 1.37 %.



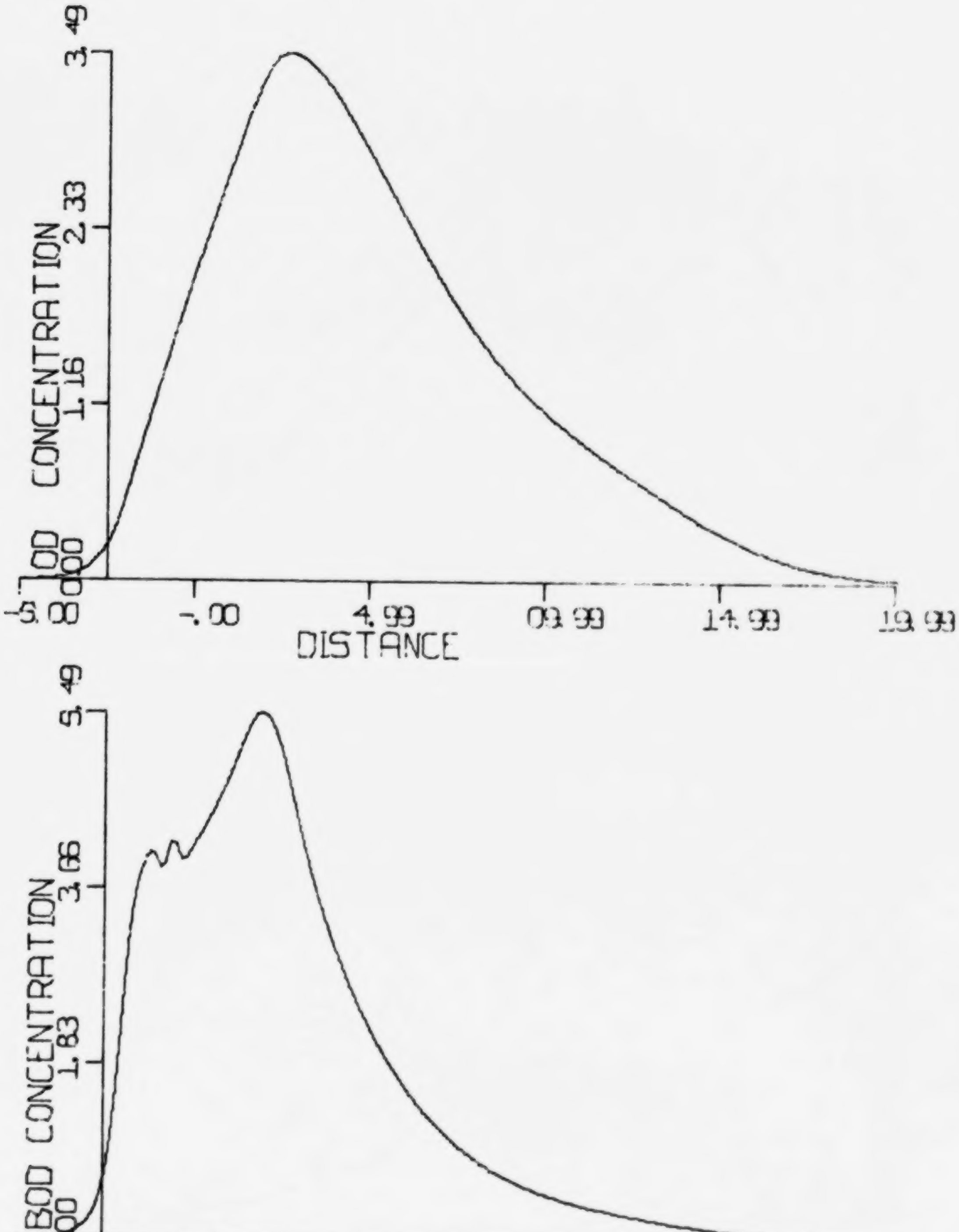
DIFFUSION MODEL - E IS ONE-HALF SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 45



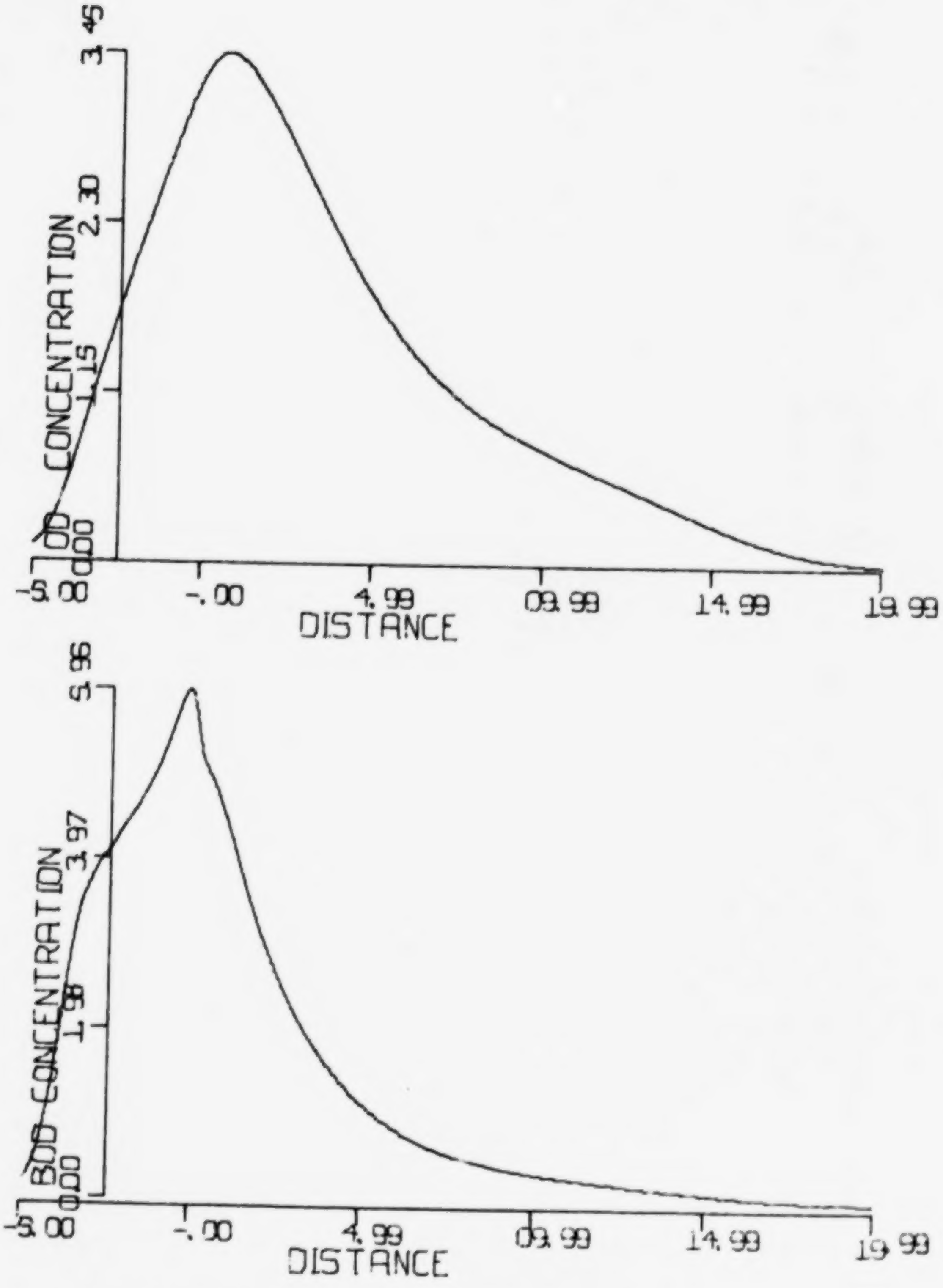
DIFFUSION MODEL - E IS ONE-HALF SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE

FIGURE 46



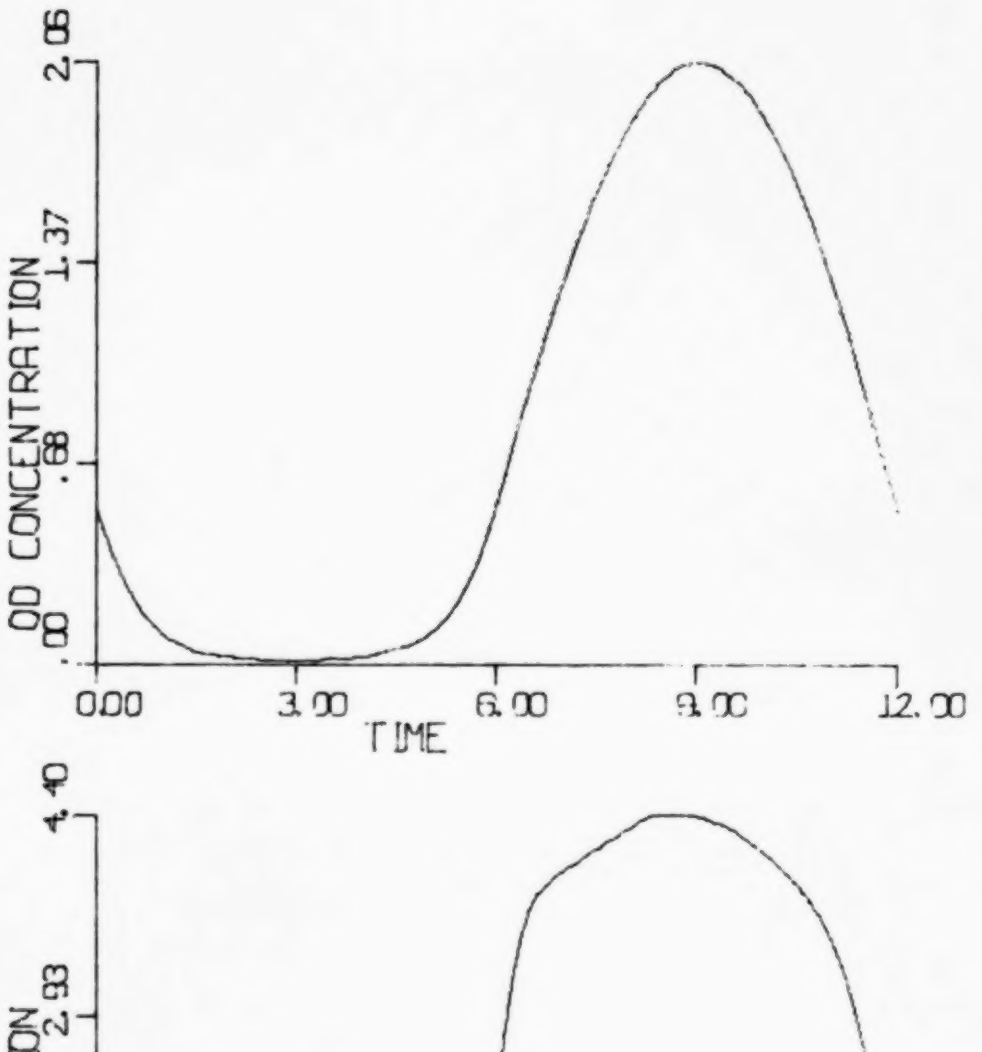
DIFFUSION MODEL - E IS ONE-HALF SQ. MI. PER DAY
B.OD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 47



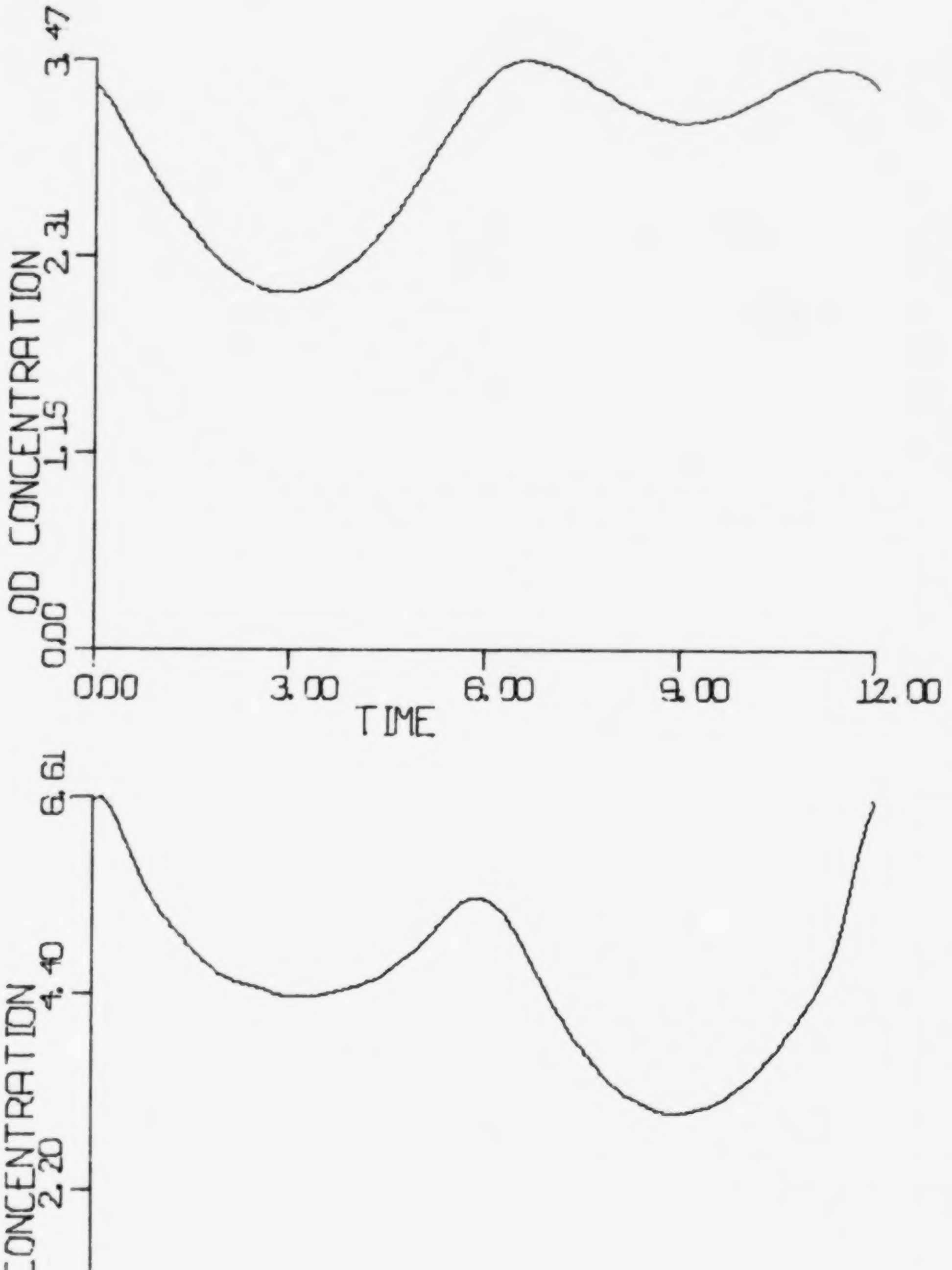
DIFFUSION MODEL - E IS ONE-HALF SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE.

FIGURE 48



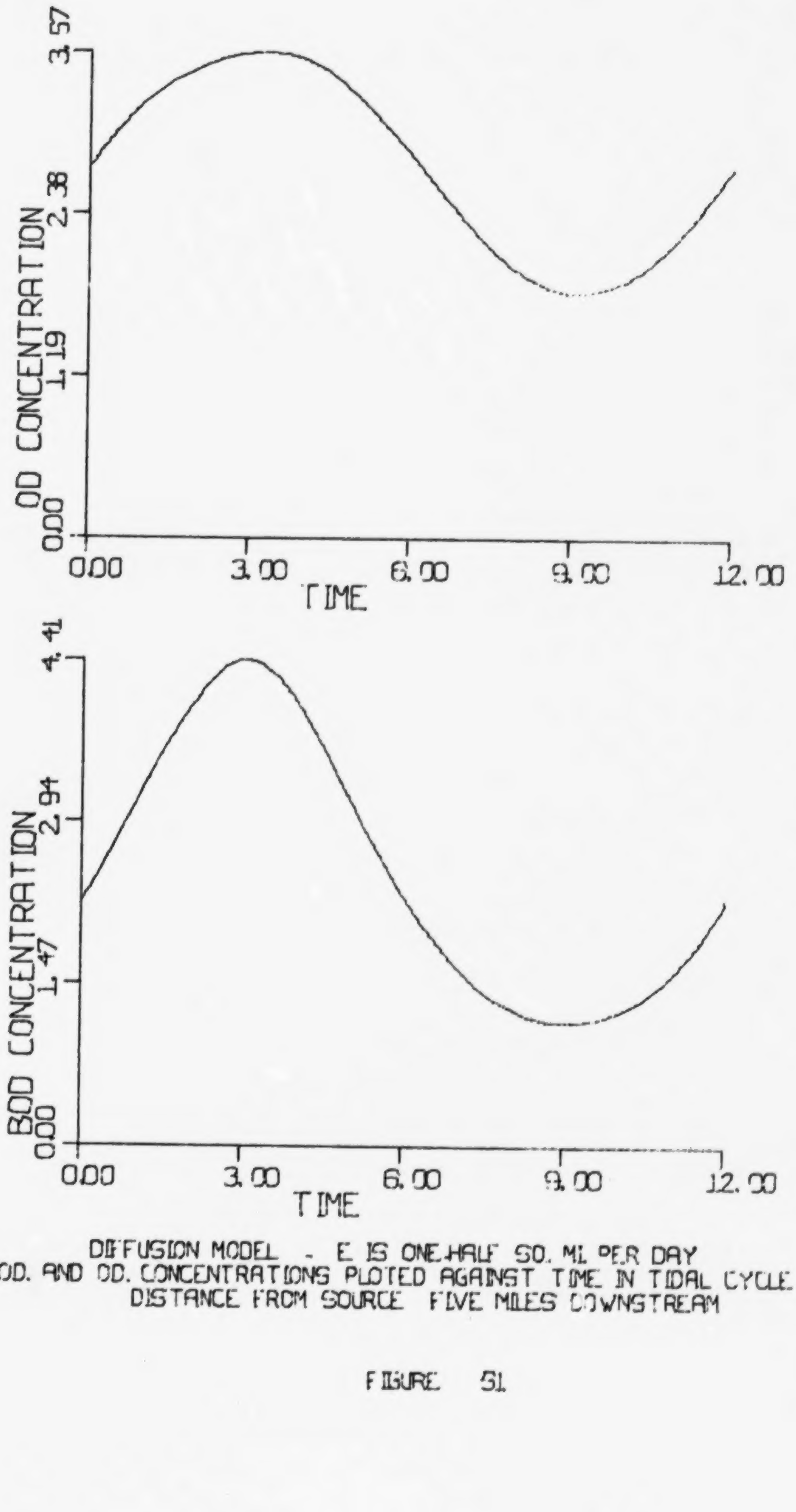
DIFFUSION MODEL - E IS ONE-HALF SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 49



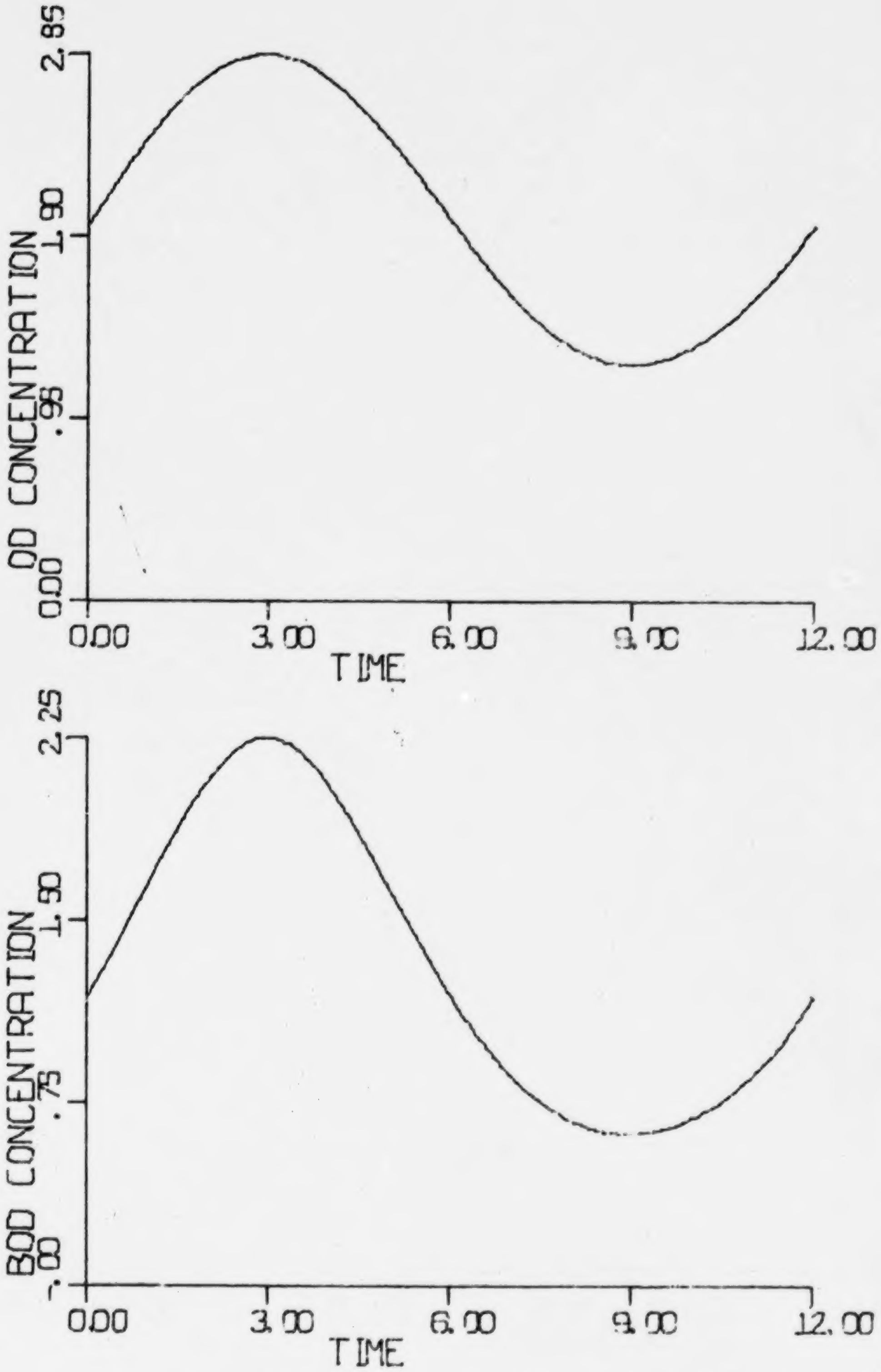
DIFFUSION MODEL - E IS ONE-HALF SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM

FIGURE 50



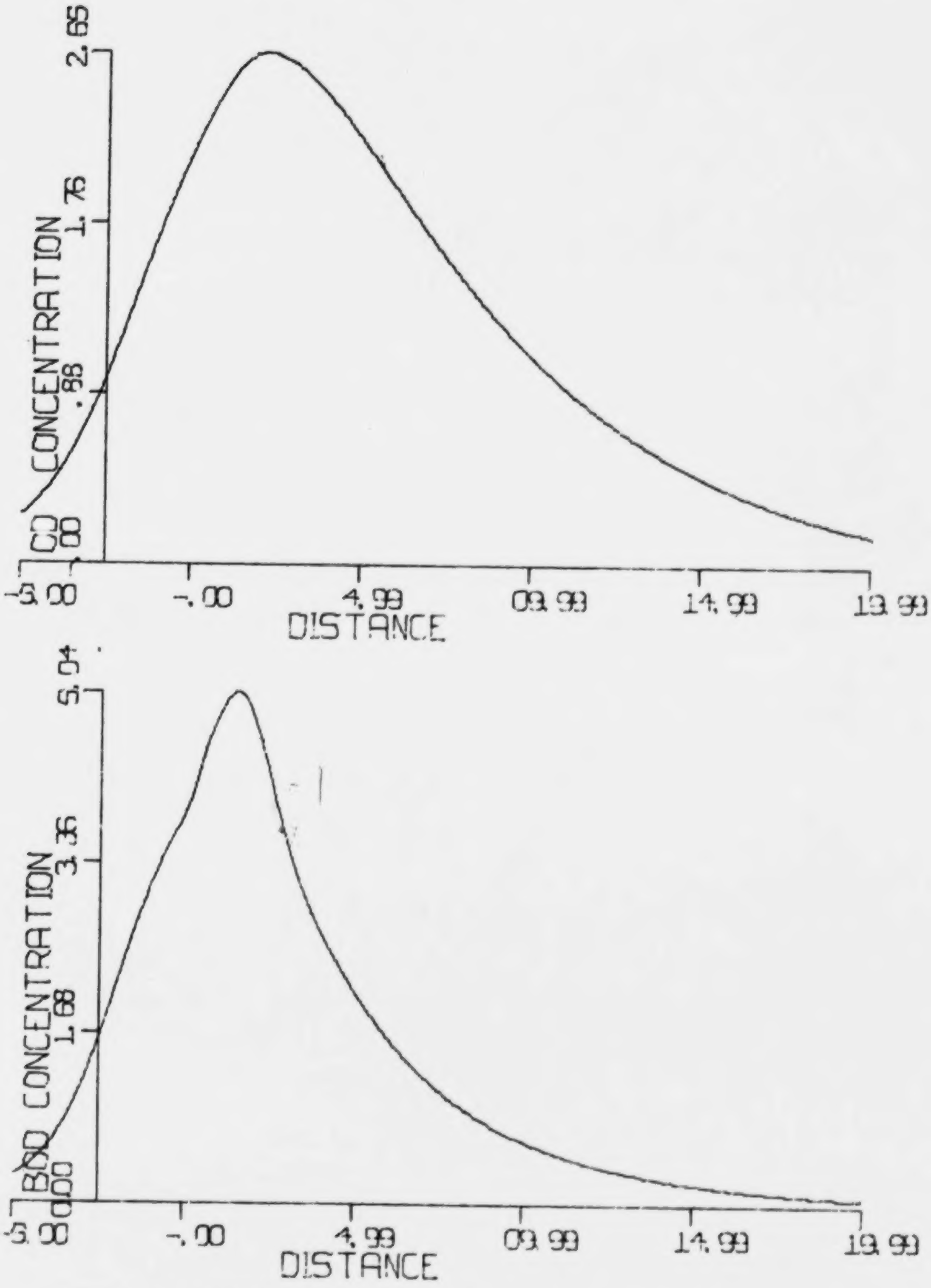
DIFFUSION MODEL - E IS ONE-HALF SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 51



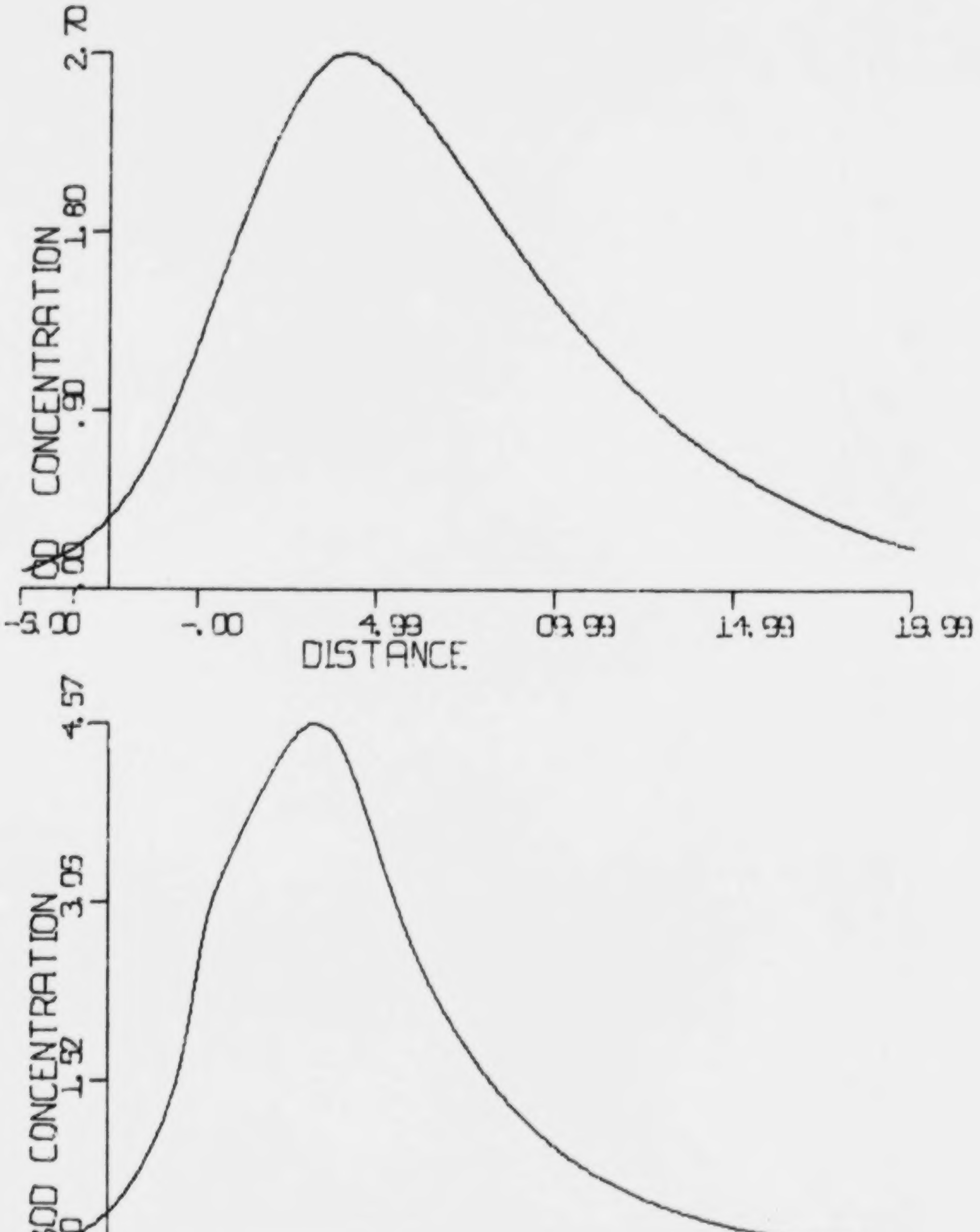
DIFFUSION MODEL - E IS ONE-HALF SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 52



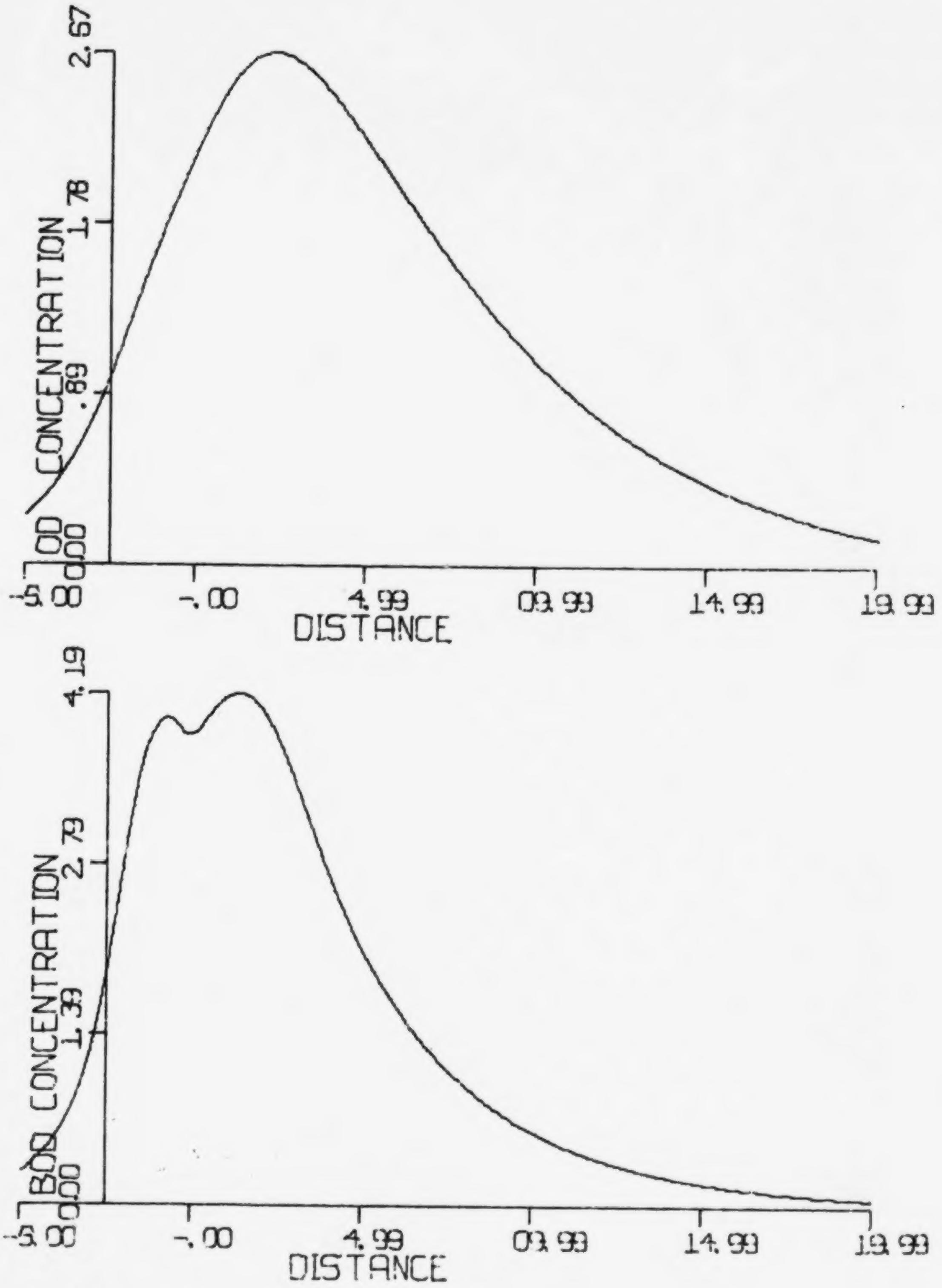
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 53



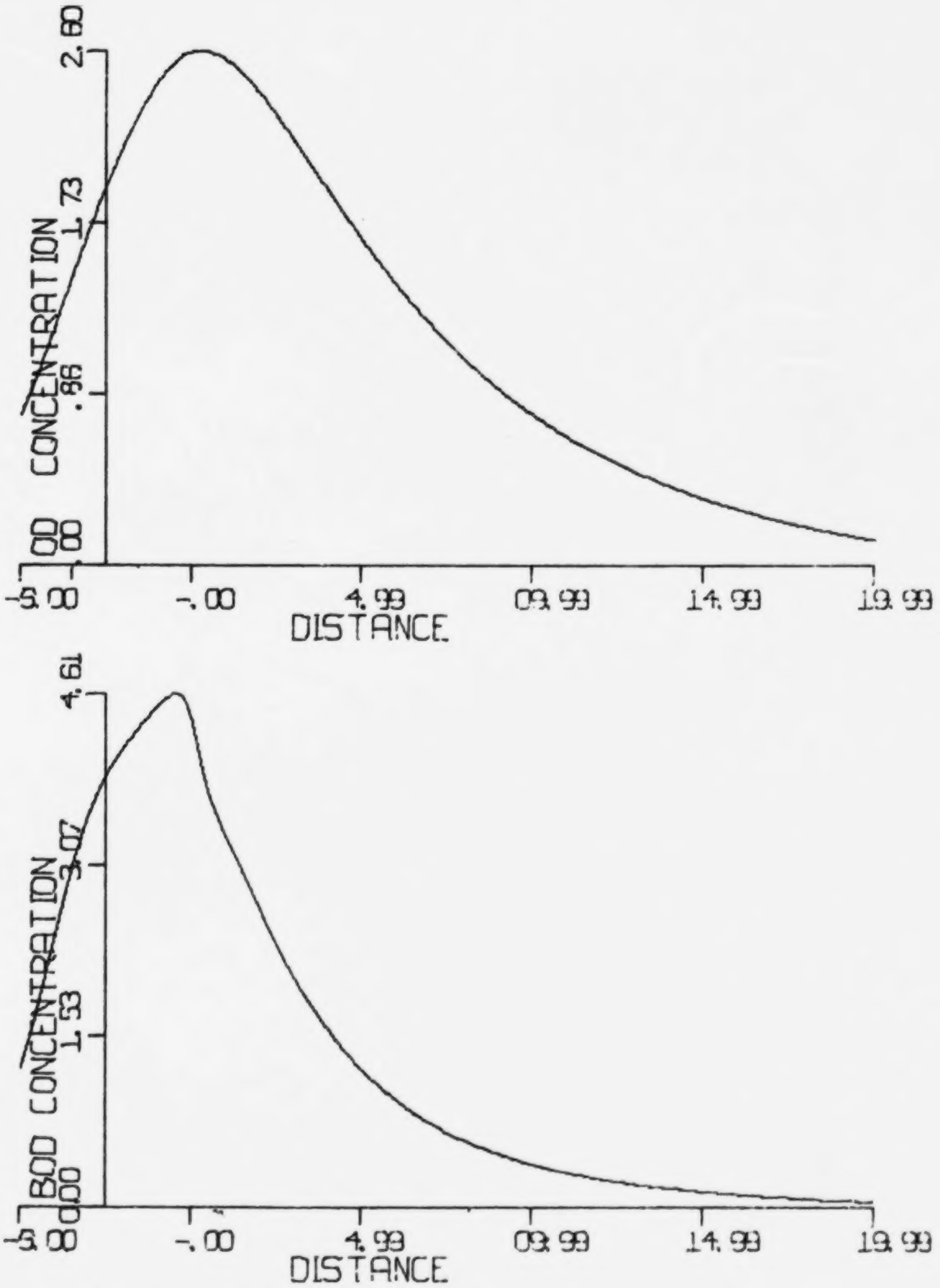
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE

FIGURE 54



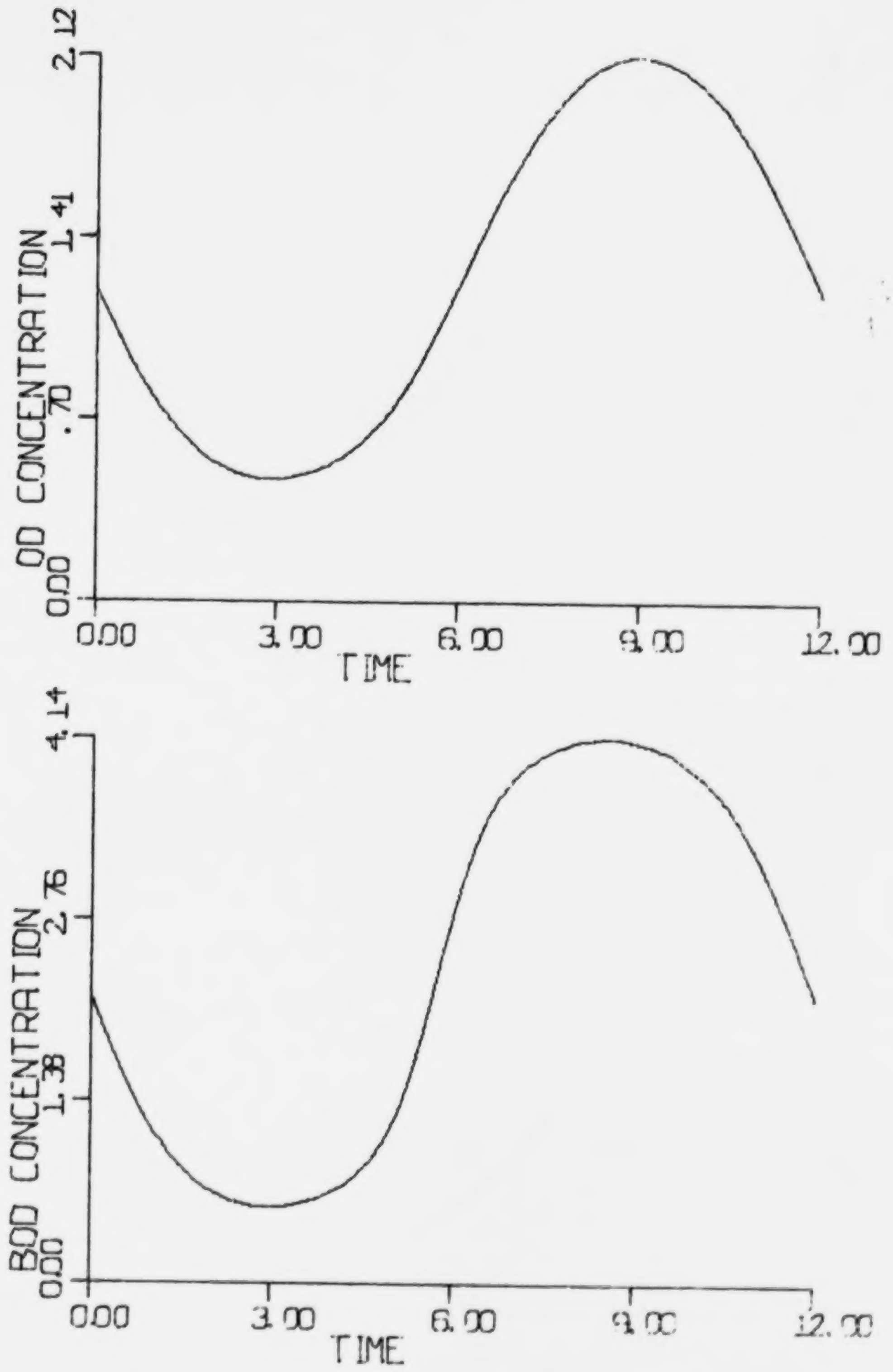
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 55



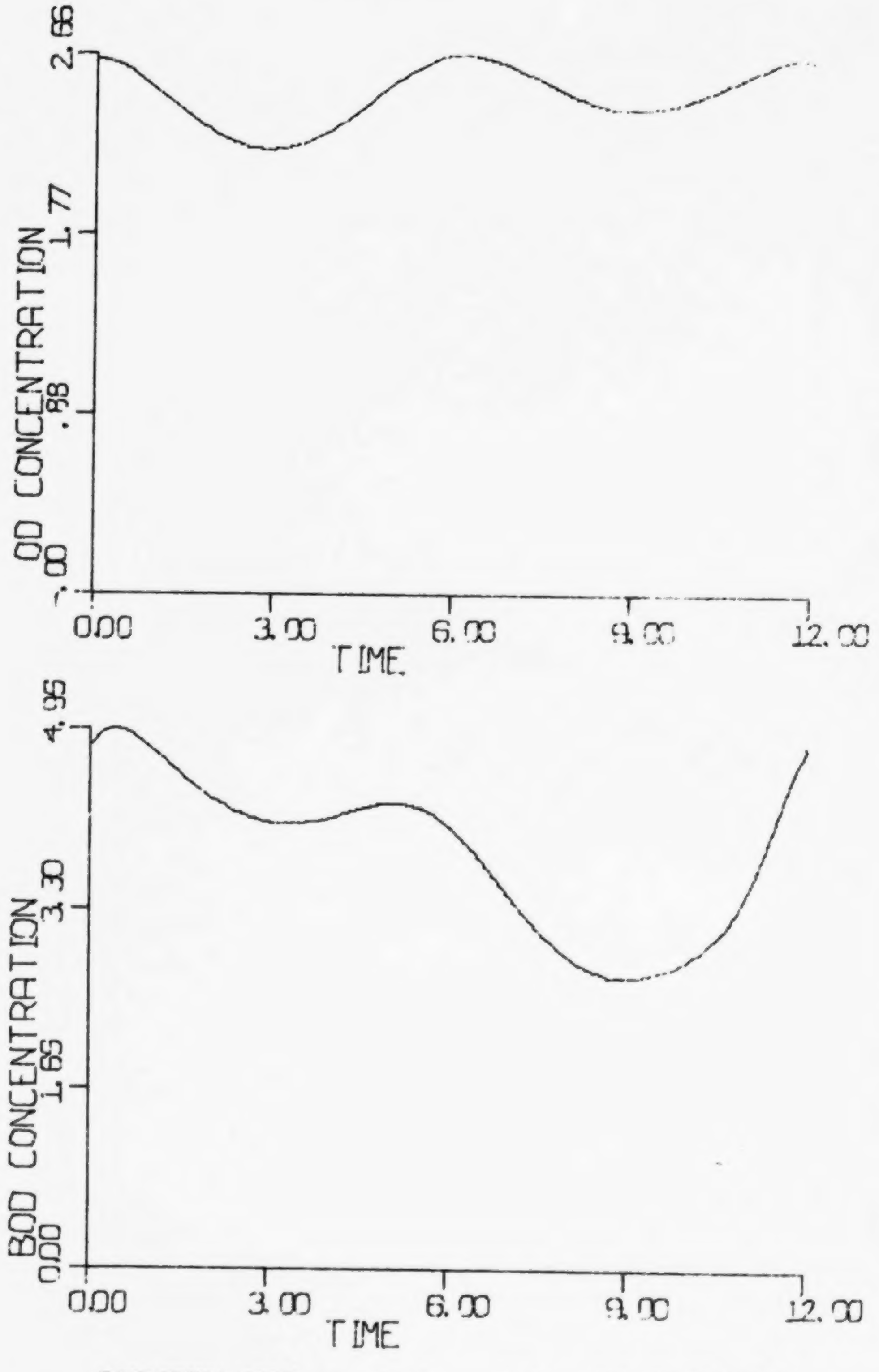
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE.

FIGURE 56



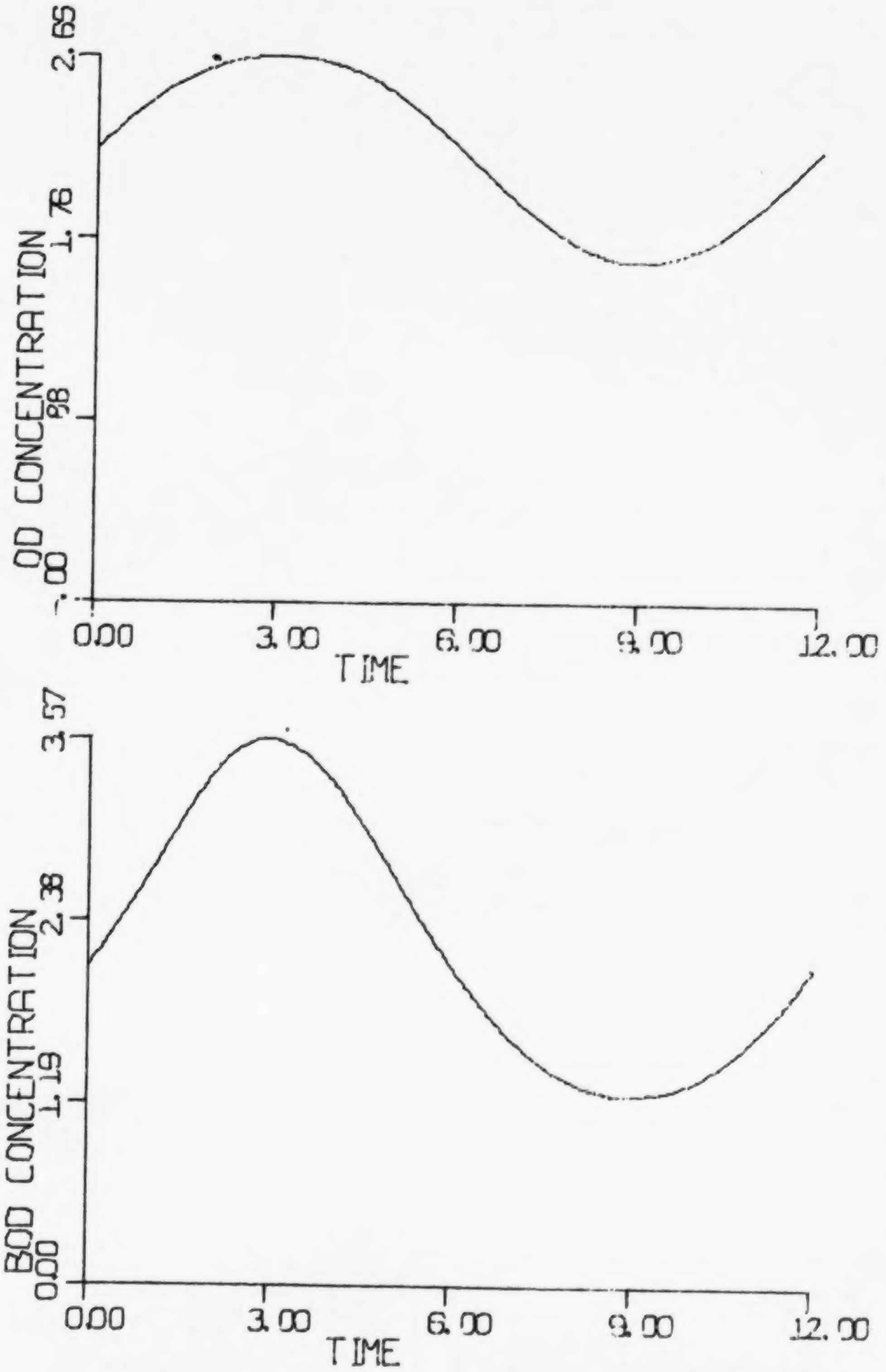
DIFFUSION MODEL - E IS TWO SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 57



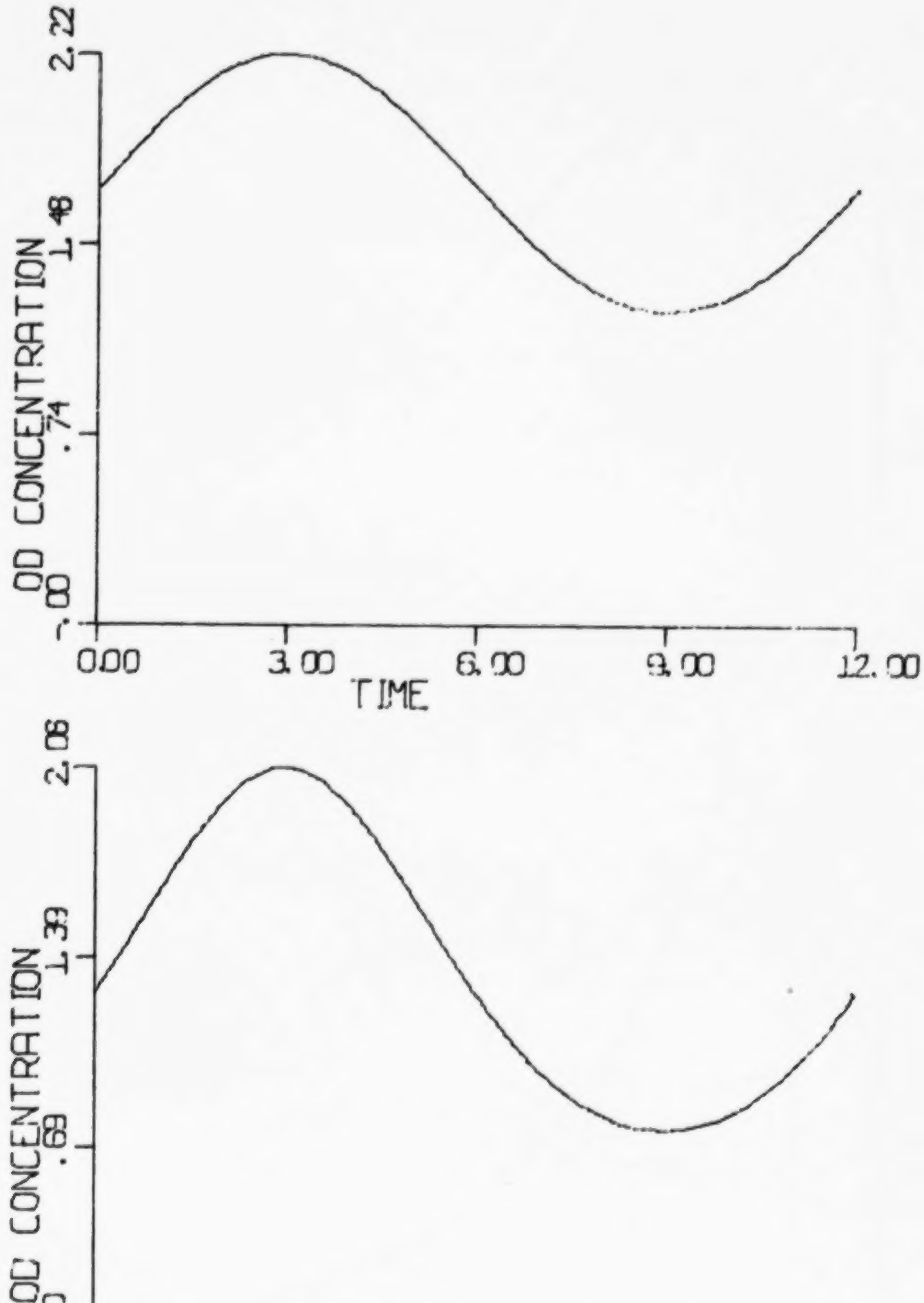
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM

FIGURE 36



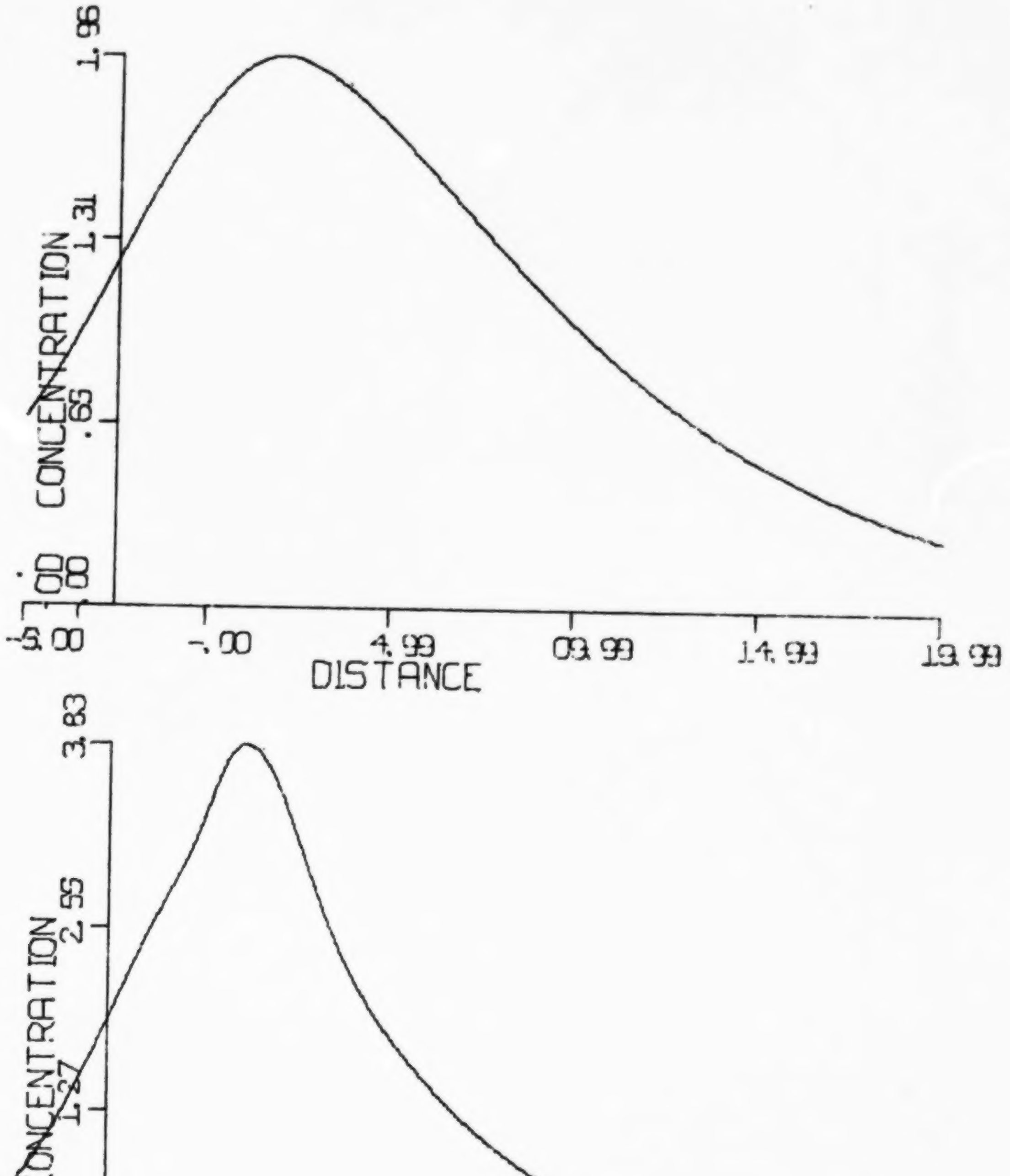
DIFFUSION MODEL - E IS TWO SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 53



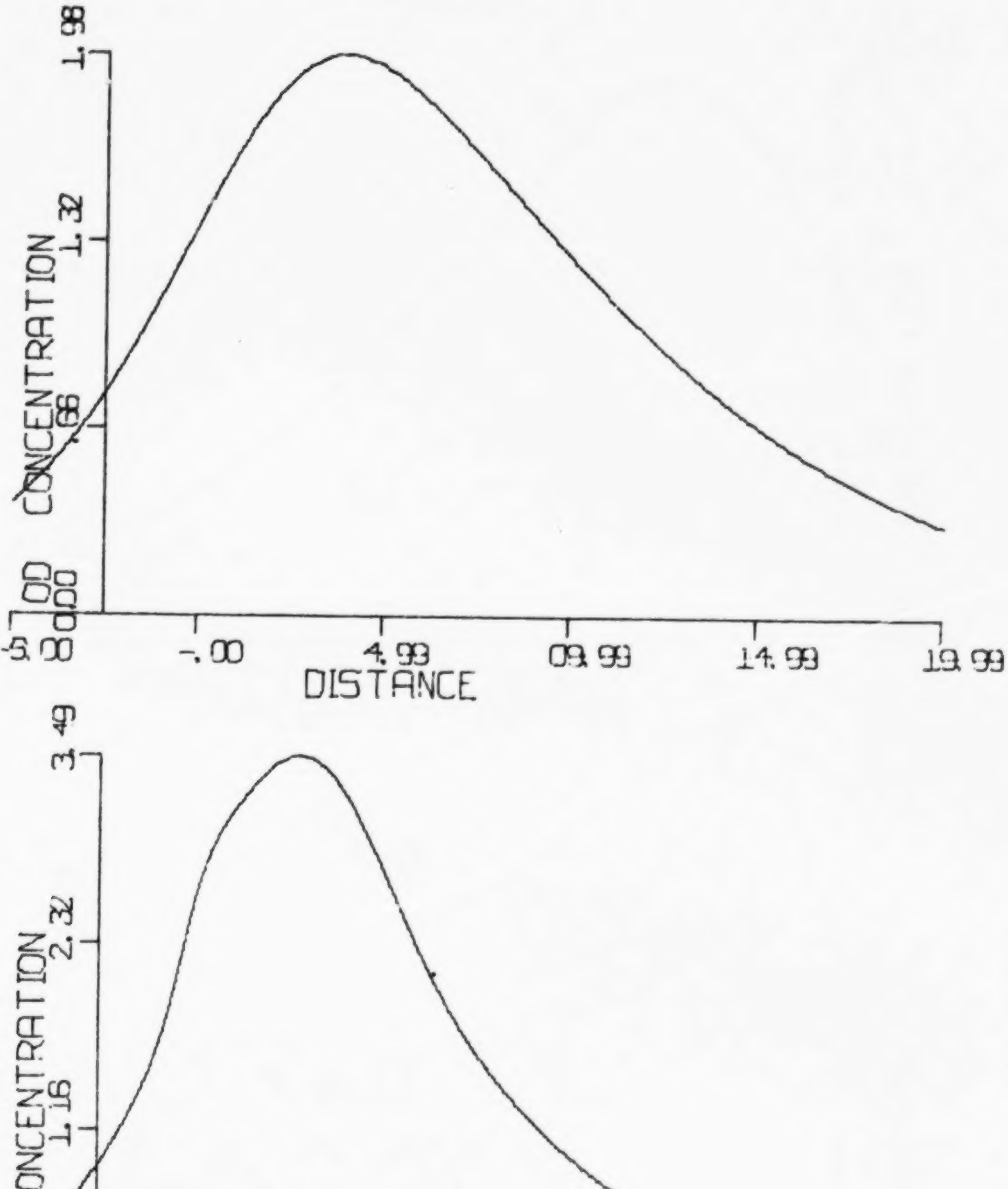
DIFFUSION MODEL - E IS TWO SQ MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 60



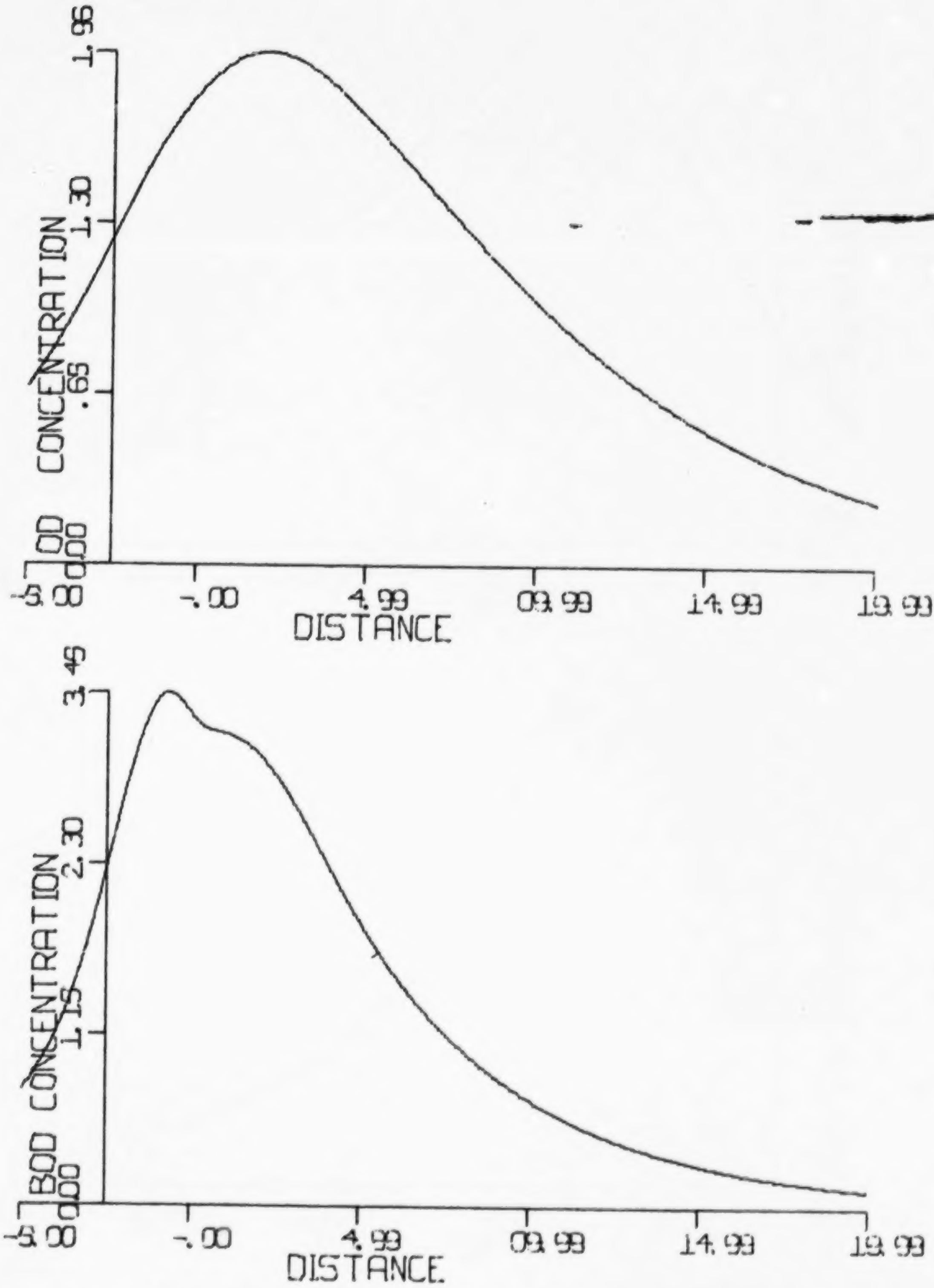
DIFFUSION MODEL - E. IS FIVE SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - POSITIVE RUSH WATER

FIGURE 61



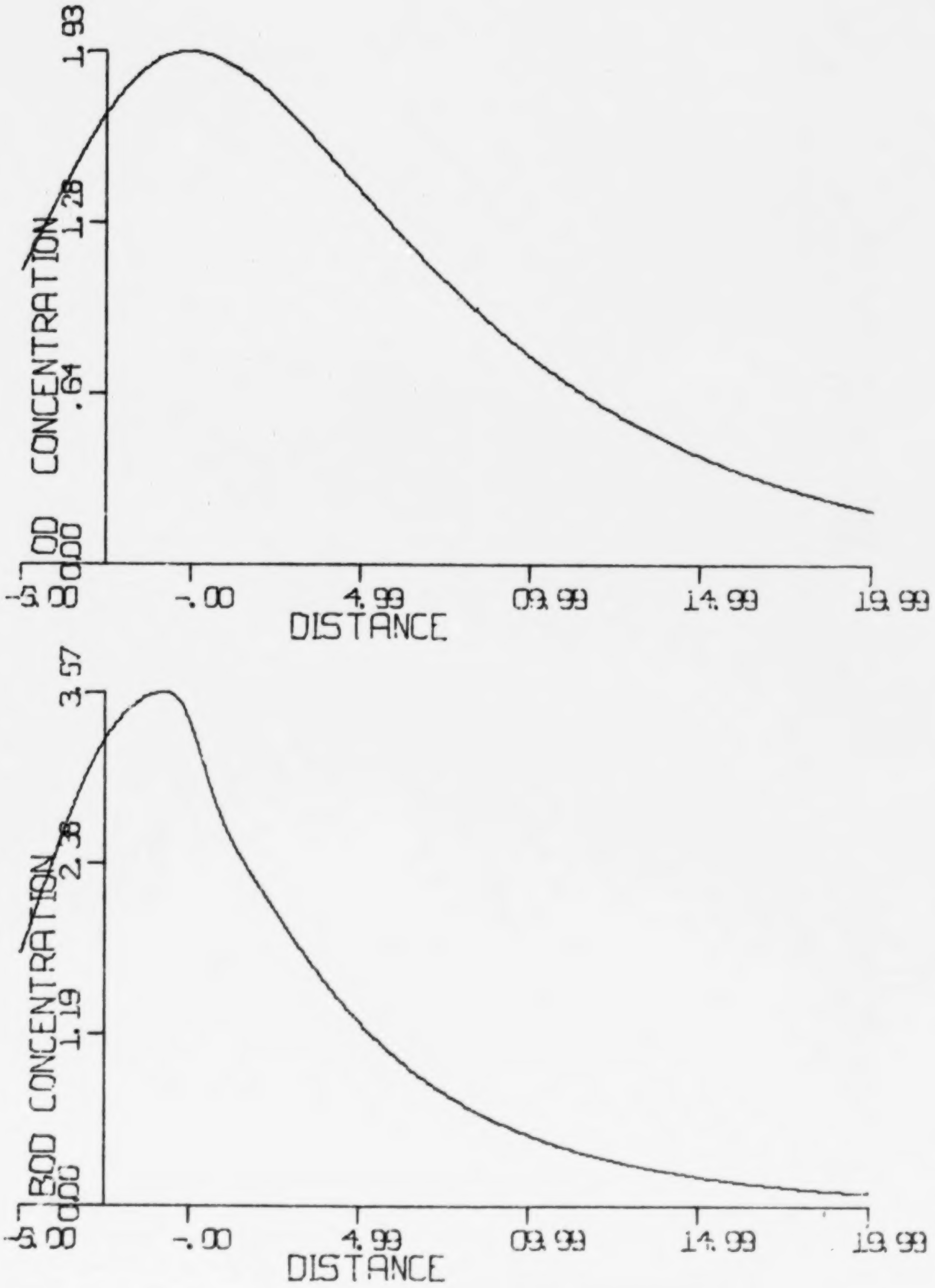
DIFFUSION MODEL - E IS FIVE SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - EBB TIDE

FIGURE 62



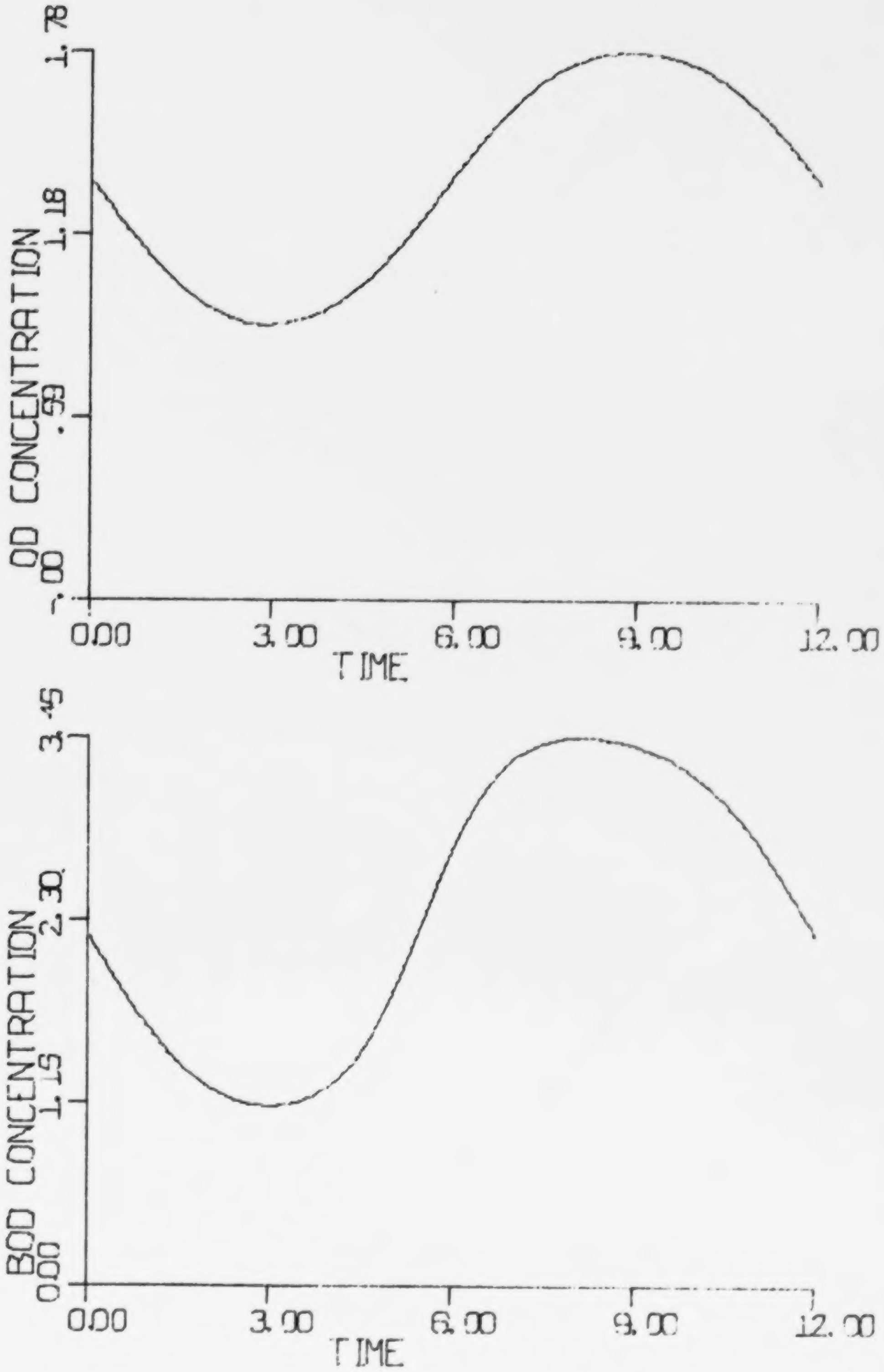
DIFFUSION MODEL - E IS FIVE SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE.
TIME IN TIDAL CYCLE - NEGATIVE RUSH WATER

FIGURE 63



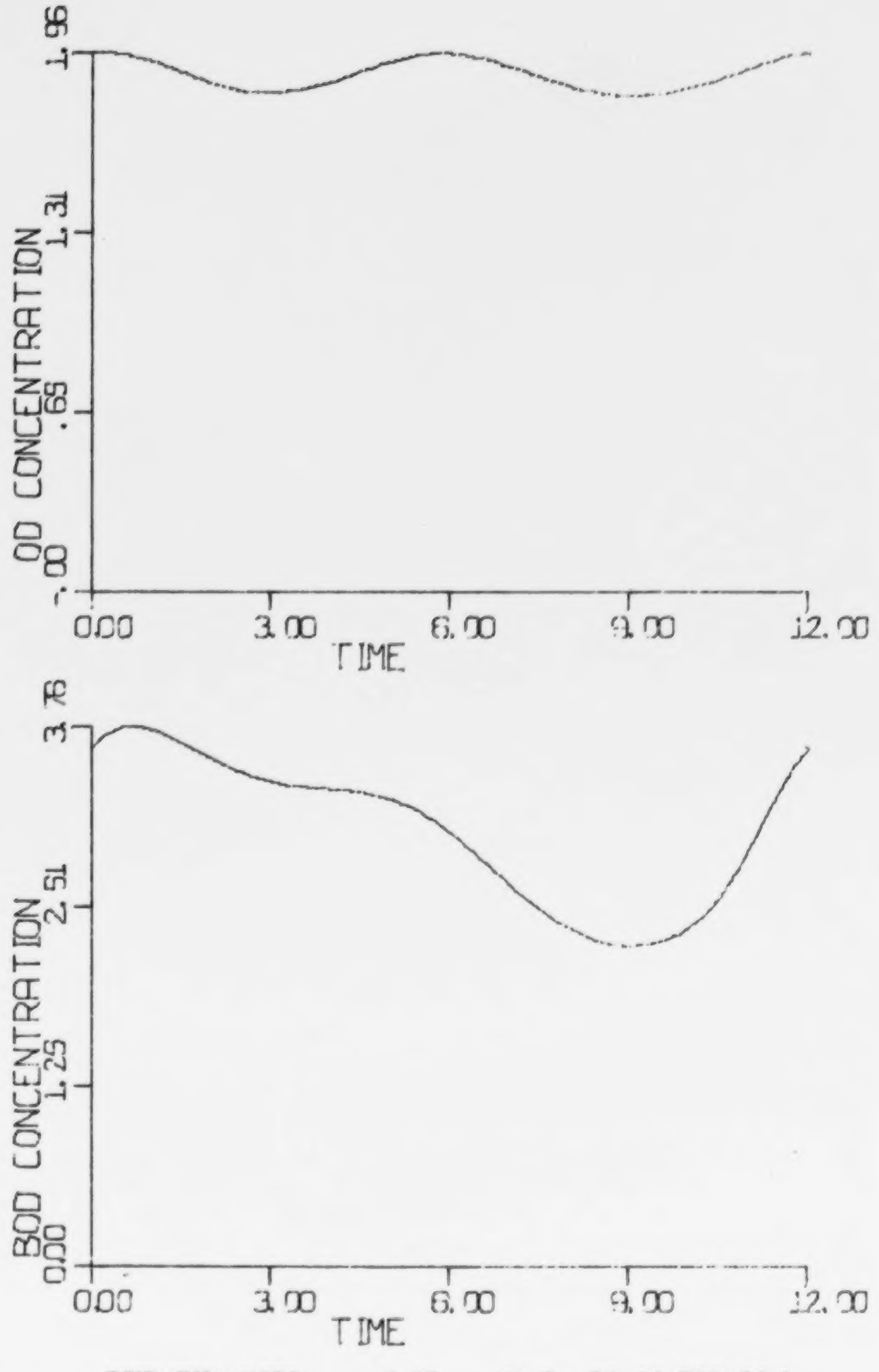
DIFFUSION MODEL - E IS FIVE SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST DISTANCE FROM SOURCE
TIME IN TIDAL CYCLE - FLOOD TIDE.

FIGURE 64



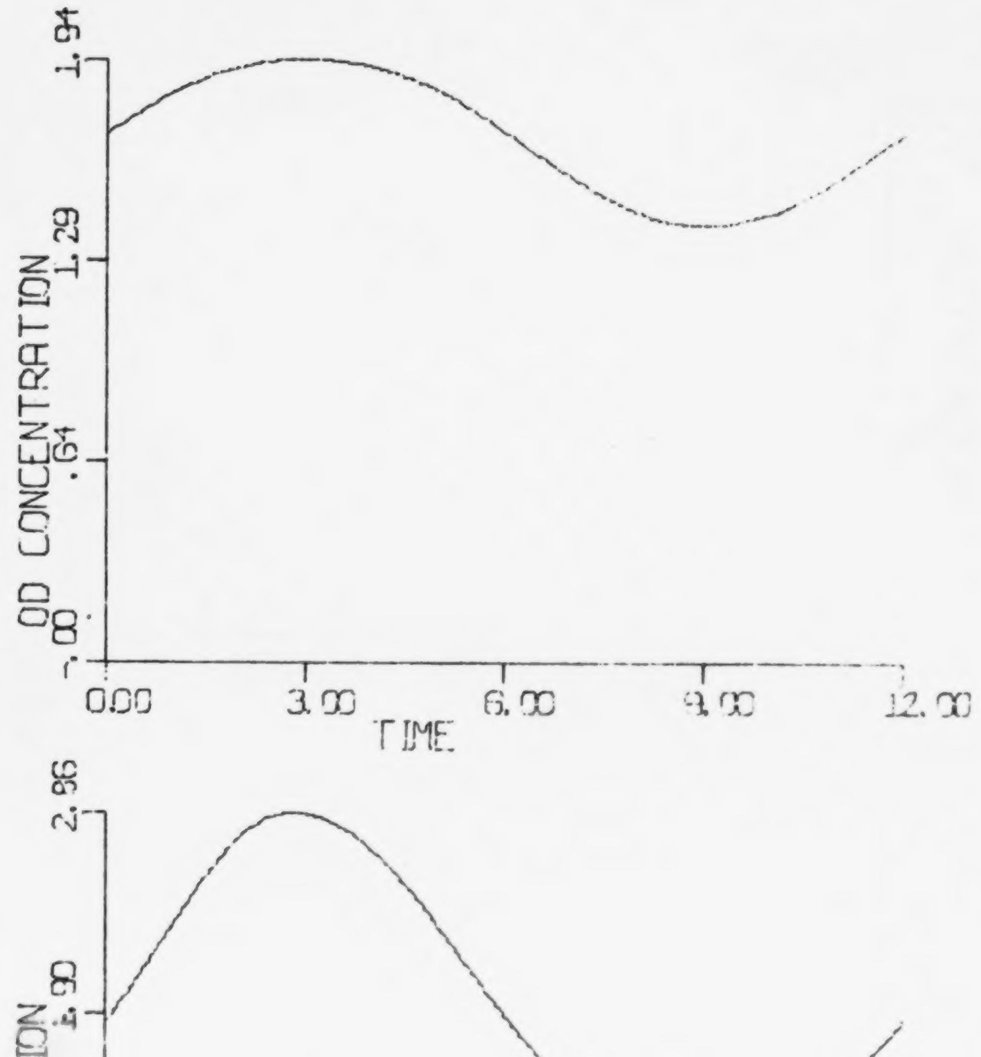
DIFFUSION MODEL - E IS FIVE SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES UPSTREAM

FIGURE 65



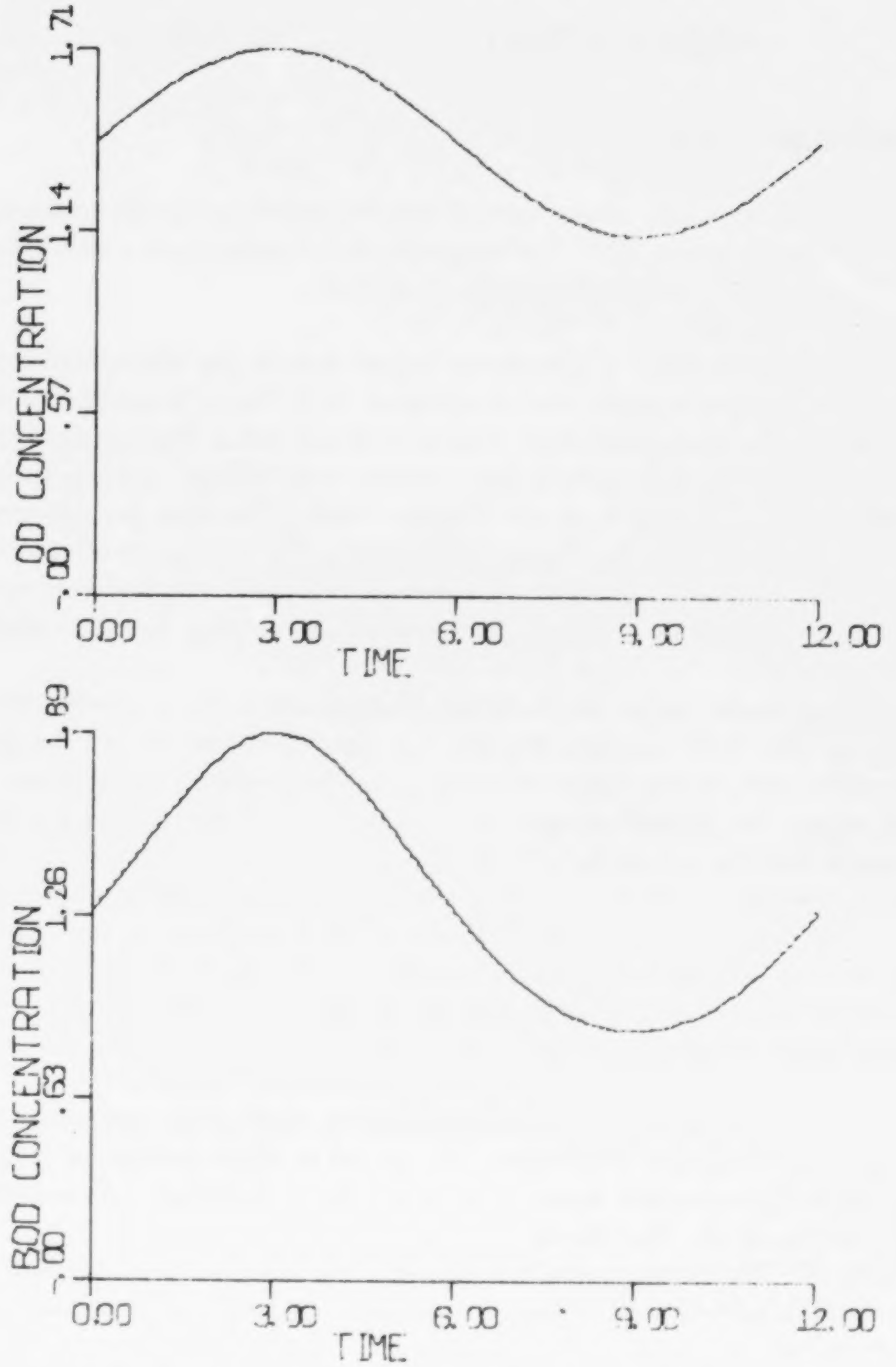
DIFFUSION MODEL - E IS FIVE SQ. MI. PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE TWO MILES DOWNSTREAM

FIGURE 66



DIFFUSION MODEL E IS FIVE SQ. MI PER DAY
BOD. AND OD. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE FIVE MILES DOWNSTREAM

FIGURE 67



DIFFUSION MODEL - E IS FIVE SQ. MI PER DAY
B.O.D. AND O.D. CONCENTRATIONS PLOTED AGAINST TIME IN TIDAL CYCLE
DISTANCE FROM SOURCE SEVEN MILES DOWNSTREAM

FIGURE 68

VERIFICATION OF DIFFUSION MODEL

The Potomac Estuary

An intensive water quality survey was conducted for the Potomac estuary during the summer of 1967. The unpublished data were made available by the Chesapeake Field Station, Annapolis, Maryland.

The Potomac River is the second largest river in the Mid-Atlantic states. The tidal portion extends from Washington, D. C. to the river's mouth which opens into the Chesapeake Bay, a distance of 116 miles. The estuary serves as the primary waste receptacle for nearly two million people living in Washington, D. C. and Northern Virginia. Table 4 describes the major waste sources on the Potomac. Figure 69 illustrates the relative locations of the sources and the water quality measurement stations. The figures represent the loadings for 1966; there was no significant change in these values for 1967.

In an estuary, pollution can remain trapped for extended periods of time, making very little progress towards the sea. Therefore, unlike the stream situation, pollution in advanced stages of decomposition may be found near the source, i.e., in the "oxygen sag" region. For this reason the Biochemical Oxygen Demand, normally used in streams, is an inadequate measure of the oxygen consuming material in an estuary. In order to account for the second stage of decomposition or "Nitrification" it is necessary to introduce the concept of Ultimate Oxygen Demand (UOD). UOD is defined as the carbonaceous oxygen demand plus the nitrogenous oxygen demand, or the total oxygen demand of the pollutant. UOD values are shown in Table 4.

In June and July of 1967, a comprehensive field survey was performed in the upper Potomac. Data was collected from four stations at half-hour intervals. The dissolved oxygen level and water temperature were recorded at each observation. The locations of the stations are shown in Table 5 and Figure 69. The survey period for Station 1 was 30 days, rendering over 1044 observations. Stations 2, 3, and 4 were observed for 14 days, yielding about 700 data points.

The relative magnitudes of the sources, and their locations relative to the stations are shown in Figure 70. The stations are conveniently located for purposes of model verification. Station 1 is upstream from the major sources and receives only back-wash effects. Stations 2 and 3, being only slightly downstream from the major source, are in the region of the "oxygen sag," while Station 4 is in the recovery region of the estuary.

Evaluation of Parameters

The method of varification is to predict the DO profiles by the use of the Diffusion Model and compare these results with the survey data. Before doing this, it is necessary to evaluate the pertinent parameters. The parameters for the section of the Potomac in question have been tabulated by Hetling and O'Connell (28). This work was completed in 1966, but should be relevant for the 1967 data. Current values of the reaeration coefficients and tidal velocities were supplied by the FWPCA Field Station at Annapolis, Maryland.

The model developed in this paper describes deoxygenation as a first order reaction. The UOD loadings given in Table 4 incorporate two separate forms of decomposition; first stage Carbonaceous Demand, and second stage Nitrogenous Demand. Thus, one must decide if different K_1 values should be used for each stage. Hetling and O'Connell recognized this problem and obtained data in order to get "a feel for the rate of nitrification." Conclusions were as follows:

> (The data) indicated that most of the nitrification occurs near the major waste treatment out-fall. Experience has indicated that this is also where most of the carbonaceous uptake occurs. Because these two reactions did appear to occur in the same portion of the estuary and since the flora and fauna involved are interrelated and in some cases are the same for each reaction, it did not appear unreasonable to use a single reaction rate constant (28).

The deoxygenation coefficient and the dissolved oxygen saturation value are functions of temperature. Hetling and O'Connell give empirical formulae as follows:

$$K_1(T) = 0.090569 - 0.009416T + 0.000879T^2 , \qquad (8.1)$$

$$C_s(T) = 14.652 - 0.4102T + 0.0079910T^2 - 0.000077774T^3, \qquad (8.2)$$

where

K_1 is the deoxygenation coefficient,

T the water temperature in degrees Centigrade,

and

C_s the DO saturation level.

In order to evaluate these coefficients, it was necessary to analyze the water temperature data. The temperature variation proved to be small. The 3517 observations ranged from 25 to 30 degrees centigrade, and 87% were between 26 and 28 degrees centigrade, inclusive. Equations (8.1) and (8.2) show only minor variations in K_1 and C_s for the variation in temperature observed. For purposes of the following analysis, K_1 was held constant at 0.48 per day, and 8.00 mg./l. was accepted as the oxygen saturation level.

The reaeration coefficient, benthal demand, and water velocity vary along the course of the estuary. The present Potomac Estuary Model segments the river, evaluating the parameter individually in each segment. Table 6 shows the parameter value for each segment. The Diffusion Model requires that these parameters remain constant in time and space. To obtain a reasonable value for the net effect of each parameter a weighted average was used. The average was obtained by using the length of the segments as weighting factors. In calculating the averages only the segments between the source and station in question were considered. Thus, a different set of parameters was used for each combination of sources and stations. Tables 7 and 8 show the values of K_2 and tidal velocity.

Equation (4.120) reveals that the mean OD level is inversely proportional to the cross-sectional area. This represents a dilution factor. The cross-sectional area used in the verification was the area at the station, since this represents the dilution at the point of measurement. These values are shown in Table 5.

No data was available for the water velocity due to fresh water flow. The position of the oxygen sag is highly sensitive to this parameter. The data from the four stations was sufficient to indicate the approximate position of the sag. It was found that a fresh water velocity of 3 mi./day would produce a sag which was consistent with the observed data. Three miles per day is a reasonable value for the portion of the Potomac under study.

The final parameter to be estimated is E, the diffusion coefficient. This is the most elusive parameter since previous studies have confounded E with tidal velocity. Dr. O'Connor estimated that the diffusion coefficient, with the tidal effect extracted, would be in the range of 2-5 mi.^{2}/day. He later commented that this estimate may be high in light of the work of Harleman and others (see Chapter III, Section B). Fortunately, the variation of the mean OD level for this range of values on E is not dramatic. For example, with all other parameters as defined above, varying E from 2 mi.^{2}/day to 5 mi.^{2}/day produced a change of about 5% in the mean OD level. The value chosen for E was 2.5 mi.^{2}/day.

Photosynthesis and Benthal Demand

Most sources of organic pollution contain a sufficient supply of nutrients to produce excessive plant growth. The photosynthetic processes of the algae growth, called algae blooms, acts as a significant oxygen source during day-light and to a lesser extent, plant respiration acts as an oxygen sink. The effect of the algae blooms is exceedingly complex, and at this time not adequately understood by sanitary engineers. Dr. Hetling commented that, "The system which controls algae blooms in the estuaries is so complex that no one has even dared to suggest that a mathematical modelof it is possible" (24).

If excessive algae are present in the estuary, marked diurnal variation in the OD concentration will be noted. To test for photosynthetic effects in the Potomac estuary, the data were grouped according to the time of day observed, and hourly averages were taken. The results are shown in Figures 71 through 74. The plots show the average OD to be significantly higher during daylight than at night. This effect is attributed to the photosynthetic action of plant life in the estuary.

The photosynthetic diurnal variation is not considered in the diffusion model. Therefore, for an unobscured verification it is necessary to remove this variation. This was accomplished considering only data points between 2:00 A.M. and 10:00 A.M. This reduced the number of available observations by two-thirds.

The above procedure successively removes the DO variation due to algae blooms, but does not remove the DO source attributed to the algae. Much of the oxygen accumulated during the day will still be present during the night. This added source of oxygen can be treated as part of the benthal demand, i.e., negative benthal demand. In order to determine the degree of the photosynthetic effect, the 2:00 A.M. to 10:00 A.M. data were averaged for each station and compared to the predicted daily averages. The predicted DO level was lower than the observed level in all cases. The difference was attributed to photosynthesis and subtracted from the benthal demand. The differences were around 1 mg./l., which is consistent with the estimates of experienced observers (28). The procedure does not prejudice the verification, because only daily averages were considered. The time dependent means were not used, and most importantly, the variation of the observed data was not considered.

Application to Potomac Estuary

The length of a tidal cycle is 12.4 hours. Hence, even though the data is restricted to the 2:00 A.M. to 10:00 A.M. period, it is uniformly distributed over the tidal cycle. In order to compare the predicted mean values with the observed data the time of observation for each data point was obtained, modulo 12.4 hours. The tidal period was segmented into 12 equal intervals, and average OD level calculated for each interval. Figures 75 through 78 show the predicted mean as a function of time, represented as a continuous curve, for each of the stations. The discrete "x's" represent the observed interval averages.

The average observed DO level at Station 3 is 0.99 mg./l. It will be recalled that the assumptions imposed during the development are violated when the DO level is exceedingly low. Special caution should be exercised in applying the predicted results to Station 3. The degree of accuracy shown in Figure 77 was unexpected, and should not give one false confidence in this region.

The important contribution of the Diffusion Model is the ability to predict the variation of the DO about the mean. To do this, it is necessary to estimate the parameter Δ. Station 1, upstream from the major pollution sources, was used as a control station. Each of the 480 data points observed at Station 1 was compared to the predicted mean. The squared error acts as an estimate of the variance at that point. It will be recalled that,

$$\sigma^2(t) = \Delta\,\mu(t) \ , \tag{8.3}$$

where $\mu(t)$ is the mean at time t, and $\sigma^2(t)$ is the variance at time t. Each of the estimates of the variance obtained by evaluating the observed squared error serves as an observation for the following linear model:

$$\sigma_i^2 = \Delta \mu i + \epsilon_i , \qquad (8.4)$$

where σi^2 is the observed variance, μi is the corresponding mean, and ϵ_i is a random error term.

Applying standard techniques, a least squares estimate of Δ was obtained. The above analysis performed on the data for Station 1 yields

$$\Delta = 0.31 \text{ mg./1.} \qquad (8.5)$$

Program 11 predicts the expected percentage of time that the OD level is above any desired value. Figures 79 through 82 plot the predicted percentages against OD concentrations. The "x's" represent the observed percentage of data points above the corresponding level. The agreement between the observed and the predicted distributions is extremely good.

In this case the peculiarity of Station 3 is noted. The predicted OD distribution has an infinite tail in the upward direction, i.e., OD levels of any positive value are possible. In field observations, the distribution is truncated at saturation. If the mean OD is sufficiently low, so that the weight in the tail exceeding saturation is negligible, no difficulty arises. In the case of Station 3 the mean OD level is 7.01 mg./l. and 36% of the observations fall above 7.5 mg./l. Thus, considerable weight is in the portion of the distribution beyond 8 mg./l., causing the relatively poor fit observed in Figure 81.

Conclusions

The variation of the observed OD values from the time averaged mean arises from two distinct sources. The first is the deterministic time dependent nature of the mean. The second is the random or stochastic fluctuation. Figures 75 through 78 illustrate the ability of the model to predict the deterministic variation; Figures 79 through 82 depict the overall variation. Inspection of the curves reveals that the model predicts the overall to a higher degree of accuracy than it predicts the time variable means. This is possible since the deterministic variation is a minor portion of the overall variation.

For example, if the predicted mean for Station 1 had coincided exactly with the observed mean, the reduction in the overall variance would only have been 5%.

Figure 82 shows that the model has slightly underestimated the variance for Station 4. Inspection of Figure 78 leads one to conclude that the error is in the estimate of the deterministic time variation, and not the stochastic component. It is apparent that increasing the magnitude of the tidal velocity will increase the time dependent fluctuations. It was conjectured that measurement errors in the tidal velocity could have produced the discrepancy. In order to test this hypothesis, the tidal velocity was increased to 35 miles per day, an increase of 25%. The noted alteration in the time dependent OD level was negligible. The maximum OD concentration increased bv slightly over 1% and the predicted minimum decreased by a like amount. The conclusion is that the predicted profile is extremely robust to deviation in the tidal velocity.

A similar comparison shows that the results were robust with respect to the diffusion coefficient. An increase of 100% (from 2.5 to 5 mi.^{2}/day) produced a change in the maximum and minimum OD values of 2.5%. Thus, it is hightly unlikely that the discrepancies between the predicted and the observed mean OD levels are due to errors in estimation of the tidal velocity of diffusion coefficient. The other parameters in the model would effect the overall mean, not the time dependent nature of the mean. The author's judgement is to attribute the prediction error to the fact that most of the parameters vary along the length of the estuary while they were considered invariant in the model.

It should be noted that the difference between the predicted and observed means is really quite small. Station 4, which is the worst situation, has a maximum error of 0.88 mg./l. and the average error is 0.44 mg./l. This degree of resolution is remarkable for a situation as dynamic as an estuary.

TABLE 4

MAJOR WASTE WATER DISCHARGES TO THE UPPER POTOMAC ESTUARY

Plant Name	Miles from Chain Bridge	Population Served	Type Treatment	UOD lbs/day
Arlington County Sewage Treatment Plant	8.9	191,000	Primary	43,400
District of Columbia Water Pollution Control Plant	10.0	1,400,000	Secondary (Activated Sludge)	318,000
Alexandria Sewage Treatment Plant	12.2	138,000	Secondary (Trickling Filter)	11,700
Fairfax County-Westgate Sewage Treatment Plant	12.3	120,000	Intermediate (Chemical Precipitation)	18,400
Fairfax County-Little Hunting Creek Sewage Treatment Plant	19.6	3,300	Secondary (Activated Sludge)	2,000
Fairfax County-Dogue Creek Sewage Treatment Plant	22.0	3,900	Secondary (Activated Sludge)	1,200

TABLE 5

WATER QUALITY MONITORS FOR THE UPPER POTOMAC

Station No.	Miles from Chain Bridge	Area Sq. Ft.	Survey Began			Survey End		
			Month	Day	Hour	Month	Day	Hour
1	4.8	24,300	6	20	1500	7	20	1800
2	11.8	47,600	7	6	1302	7	20	1802
3	17.8	52,000	7	6	1304	7	20	1804
4	29.5	80,000	7	6	1306	7	20	1806

TABLE 6

POTOMAC ESTUARY PHYSICAL PARAMETERS

Segment	Miles from Chain Bridge	Length of Segment	K_2 1/Day	V_T Mi./Day	D_B Mg./L./Day
1	0.00	2.69	0.05	9.27	0.00
2	2.69	2.09	0.09	10.61	0.04
3	4.78	1.75	0.17	11.55	0.39
4	6.53	1.31	0.37	13.64	0.79
5	7.84	1.87	0.31	19.32	0.71
6	9.71	2.11	0.39	25.00	0.91
7	11.82	2.57	0.55	26.89	1.01
8	14.39	2.19	0.32	27.46	0.73
9	16.58	2.44	0.25	28.03	0.67
10	19.02	2.06	0.32	28.59	0.55
11	21.08	2.25	0.23	28.79	0.42
12	23.33	3.95	0.26	28.79	0.43
13	27.28	2.94	0.26	26.69	0.36
14	30.22	2.93	0.28	23.29	0.40
15	33.15	4.41	0.33	22.16	0.45

TABLE 7

MEAN K_2 VALUES BETWEEN STATIONS AND SOURCES

Units 1/Day

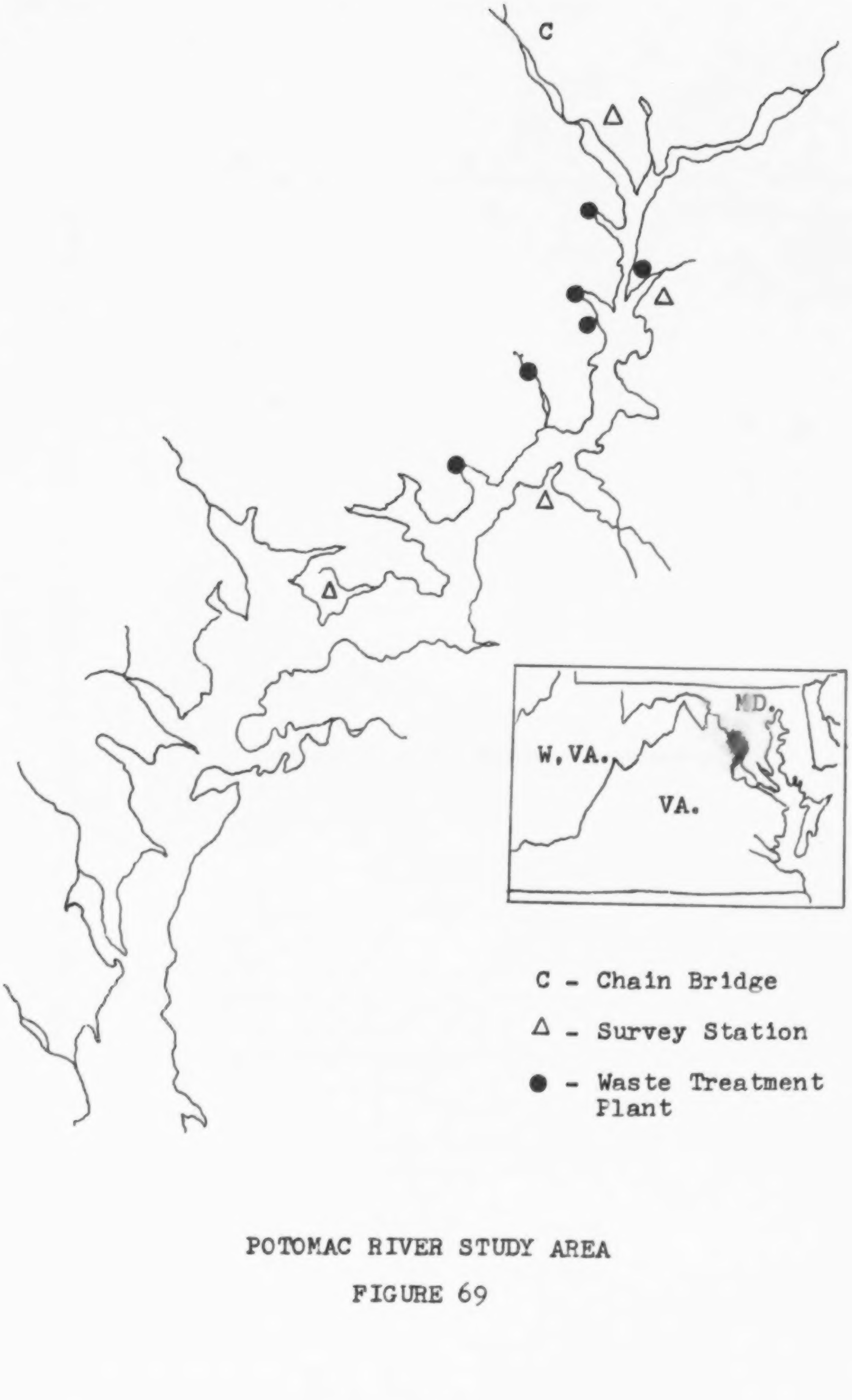
Station Source	1	2	3	4
A	0.28	0.43	0.37	0.32
B	0.31	0.49	0.38	0.32
C	0.37	0.55	0.38	0.31
D	0.37	0.55	0.38	0.31
E	0.35	0.38	0.30	0.27
F	0.33	0.35	0.28	0.25

TABLE 8

MEAN V_T VALUES BETWEEN STATIONS AND SOURCES

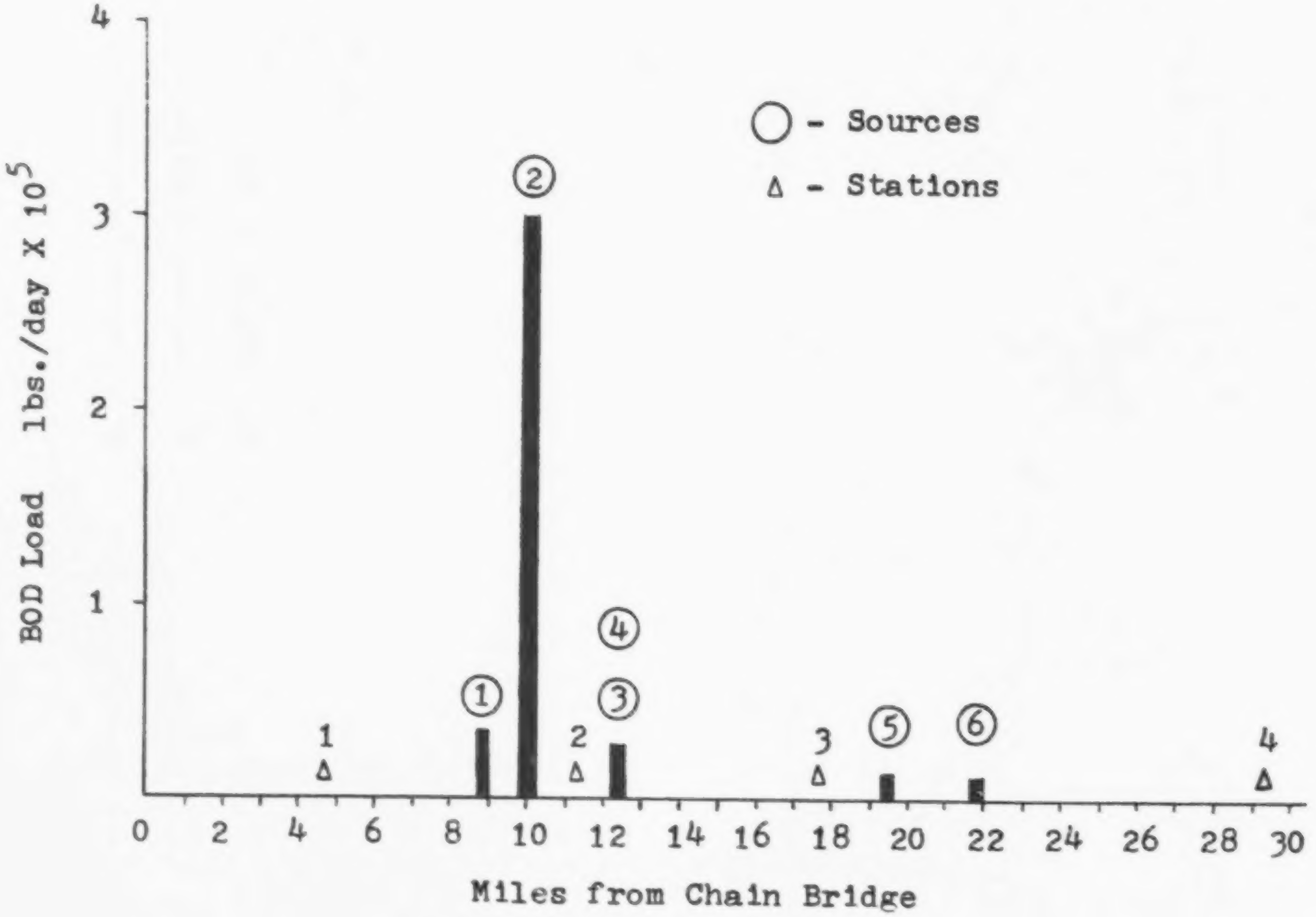
Units Mi./Day

Station Source	1	2	3	4
A	14.96	24.05	25.57	26.89
B	17.99	26.52	26.89	27.65
C	20.45	26.89	27.47	27.84
D	20.45	26.89	27.46	27.84
E	23.48	27.65	28.22	28.22
F	24.24	27.84	28.41	28.03


C
W,VA.
MD.
VA.
C - Chain Bridge
△ - Survey Station
● - Waste Treatment Plant

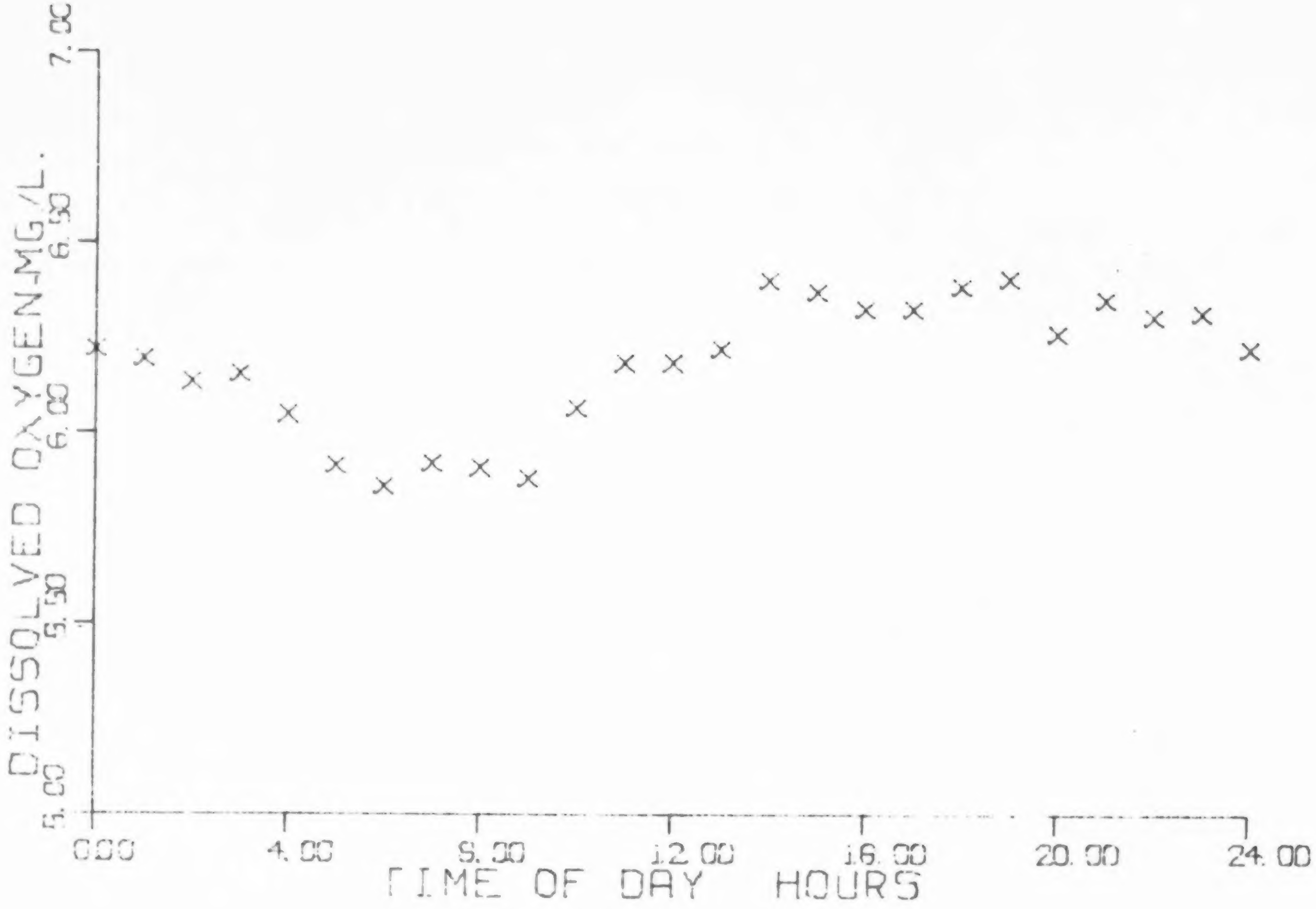
POTOMAC RIVER STUDY AREA

FIGURE 69



LOCATIONS OF SOURCES AND STATIONS, MAGNITUDE OF SOURCES

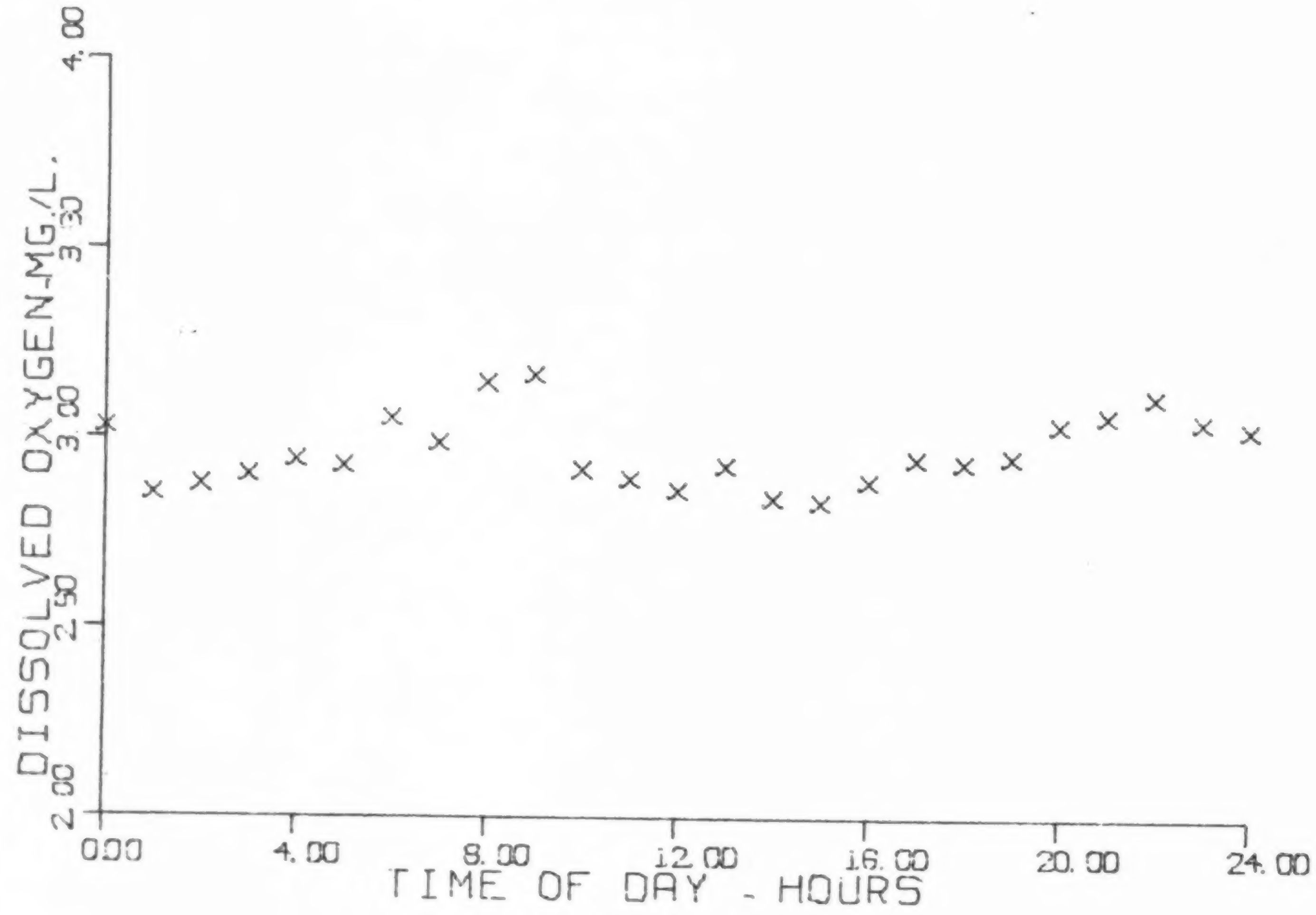
FIGURE 70



STATION ONE

OBSERVED HOURLY AVERAGE DISSOLVED OXYGEN LEVELS

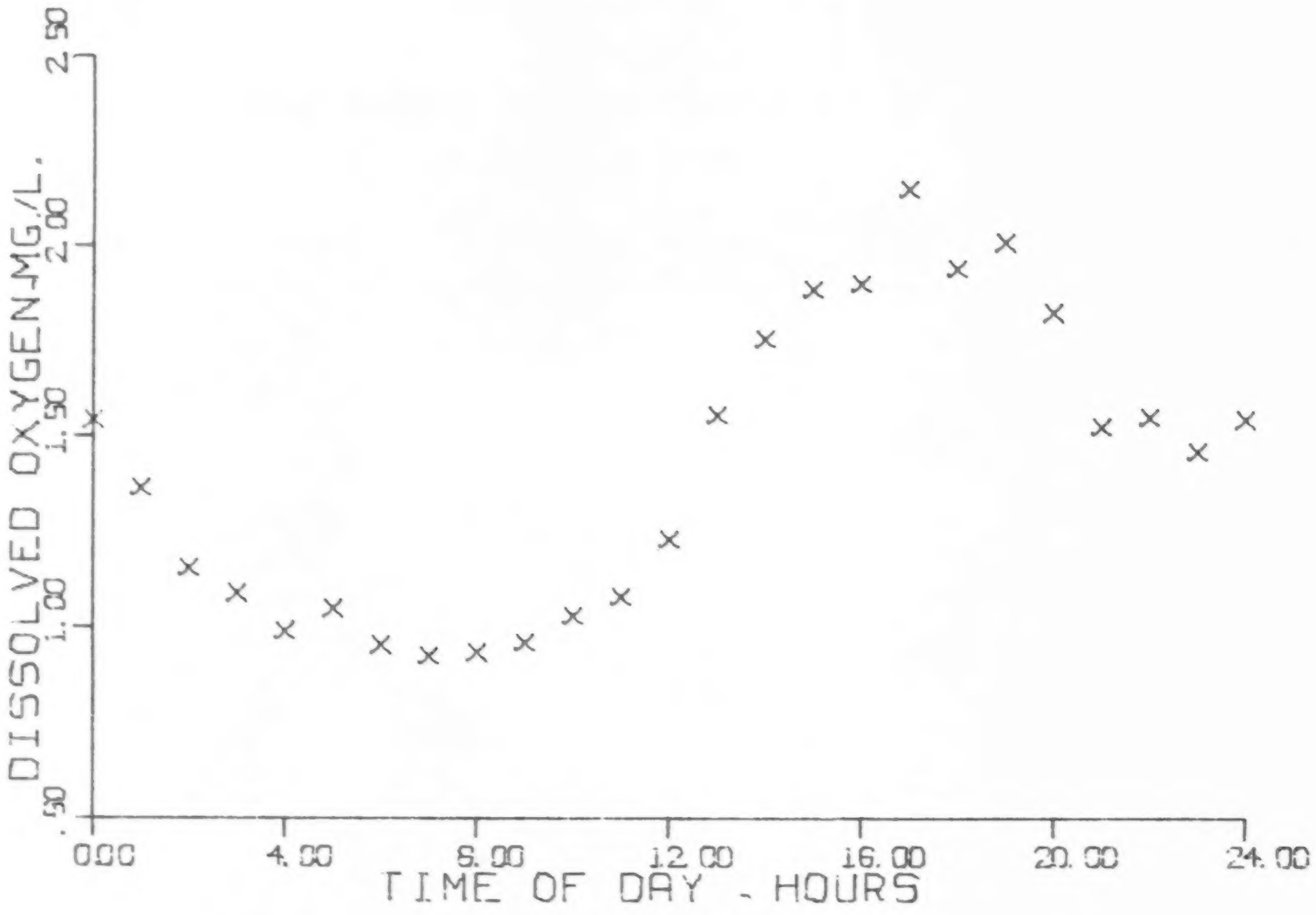
FIGURE 71



STATION TWO

OBSERVED HOURLY AVERAGE DISSOLVED OXYGEN LEVELS

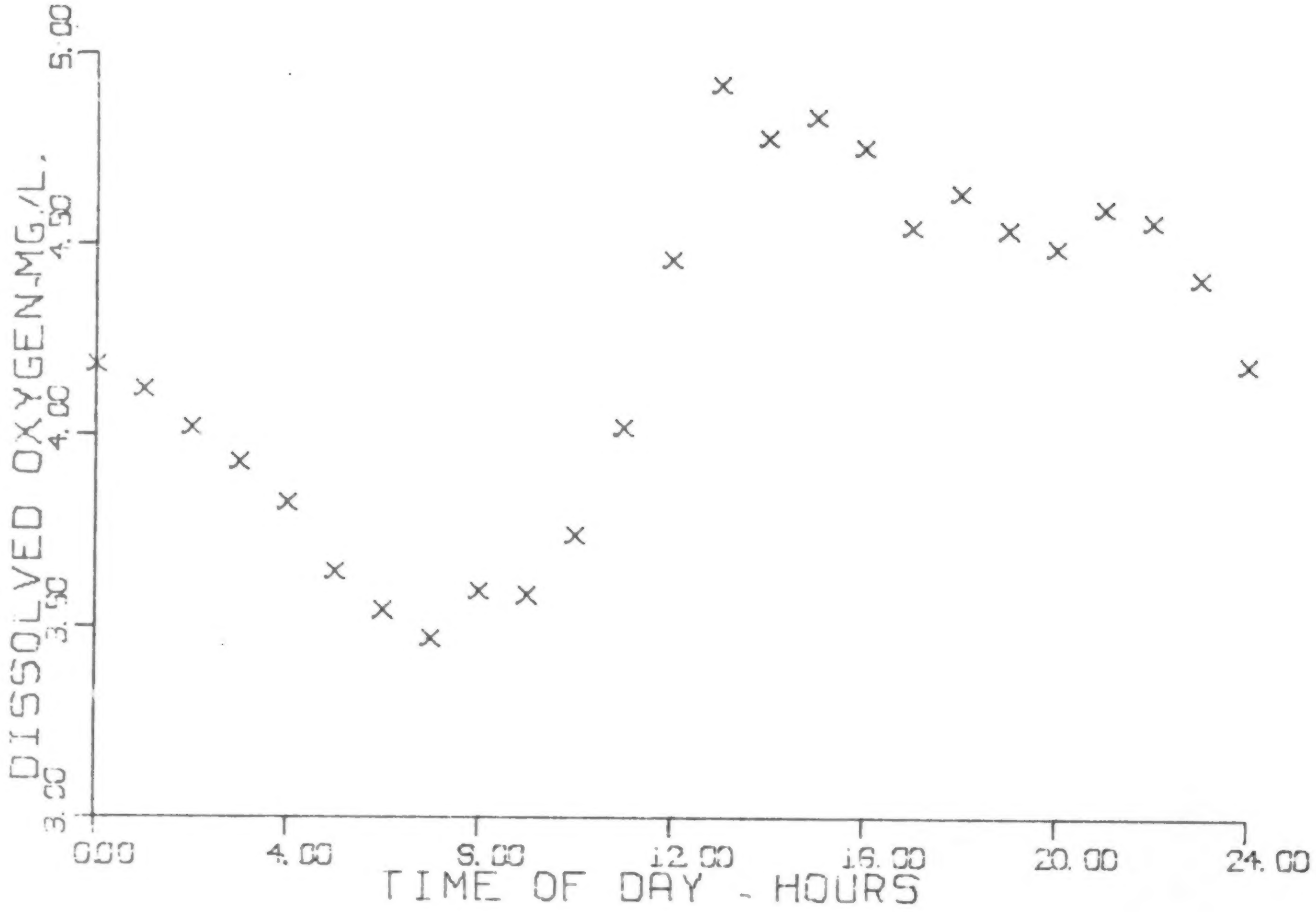
FIGURE 72



STATION THREE

OBSERVED HOURLY AVERAGE DISSOLVED OXYGEN LEVELS

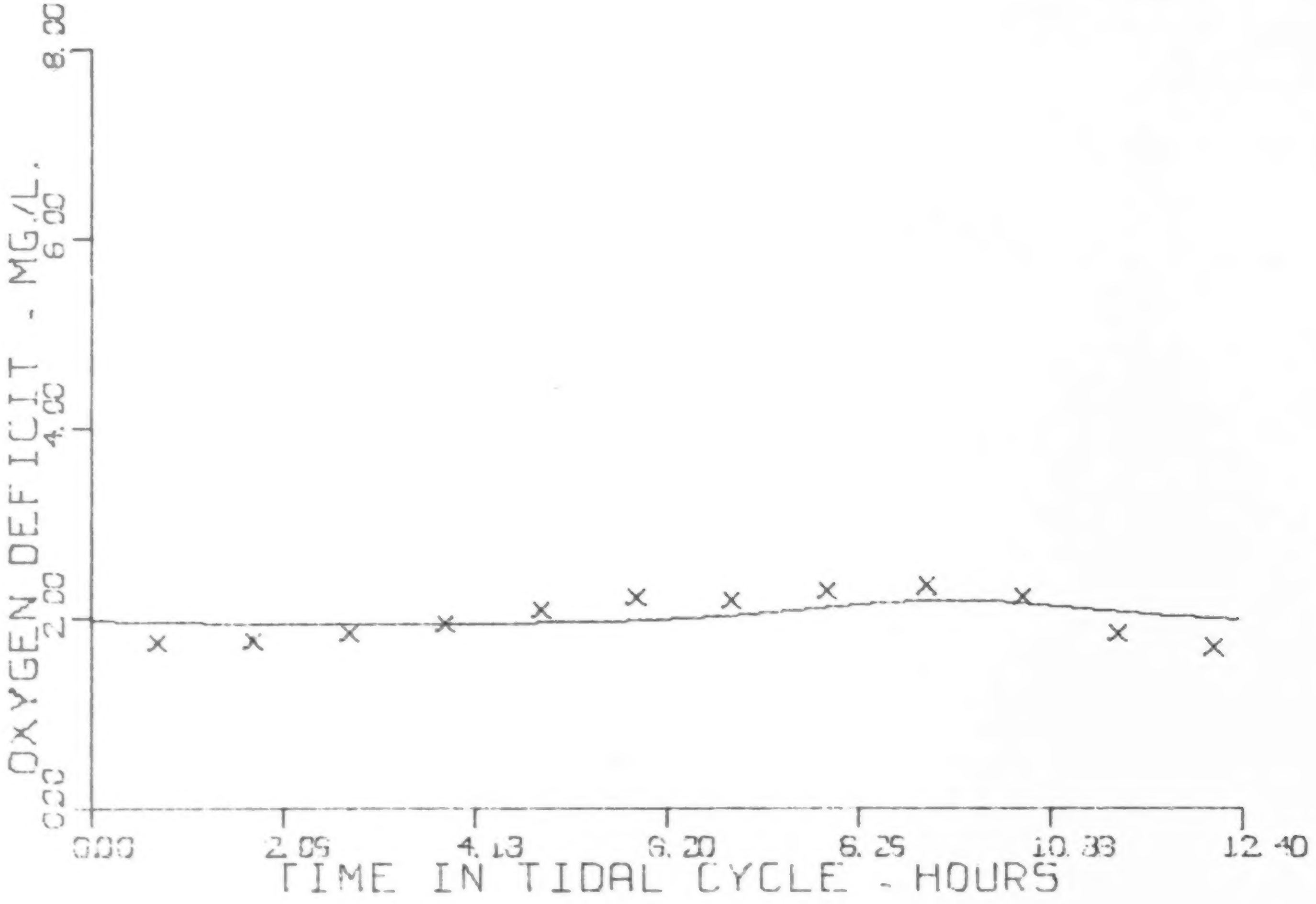
FIGURE 73



STATION FOUR

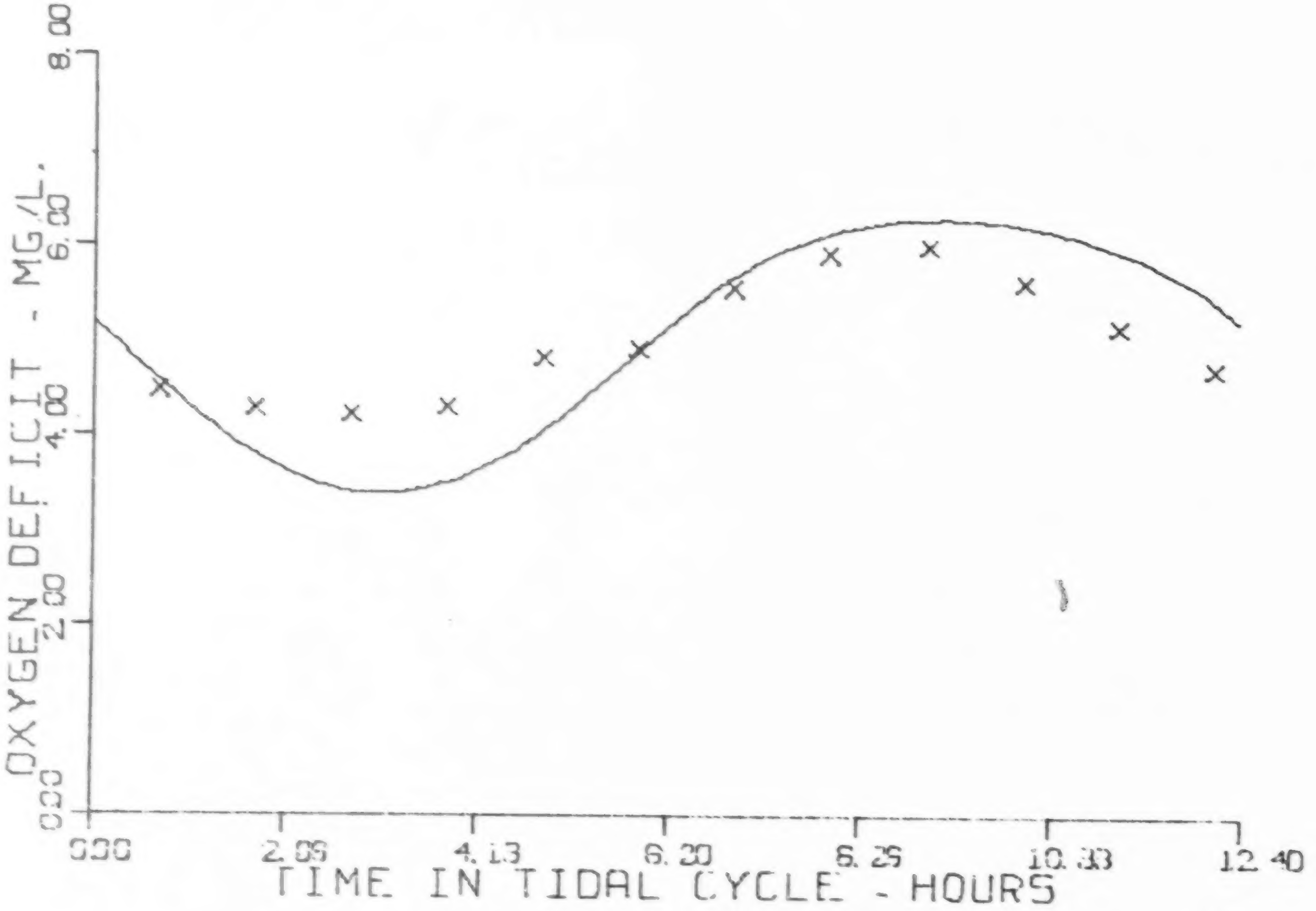
OBSERVED HOURLY AVERAGE DISSOLVED OXYGEN LEVELS

FIGURE 74



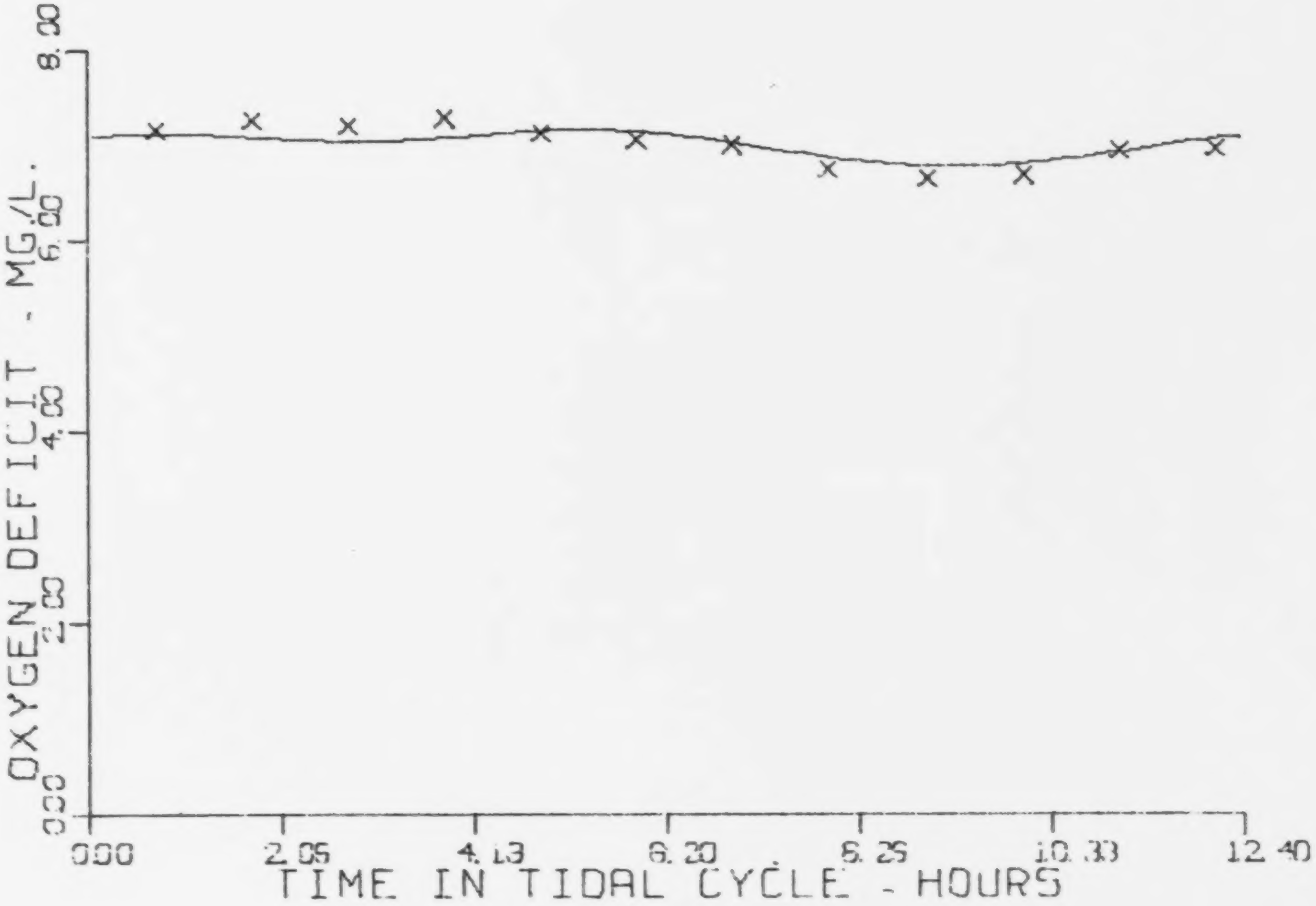
STATION ONE
PREDICTED MEAN OD LEVEL AND OBSERVED HOURLY AVERAGE OD LEVELS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 75



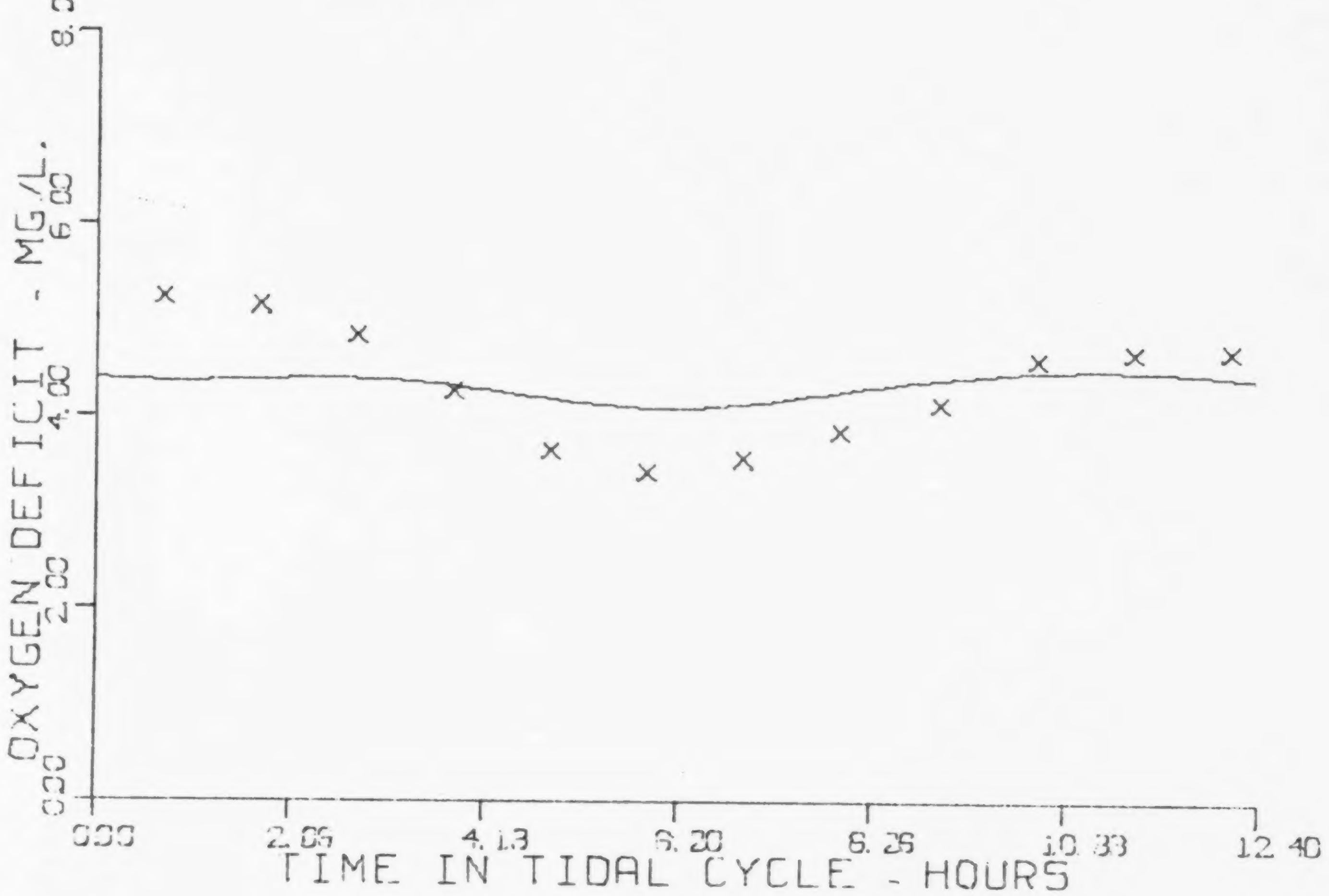
STATION TWO
PREDICTED MEAN OD LEVEL AND OBSERVED HOURLY AVERAGE OD LEVELS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 79



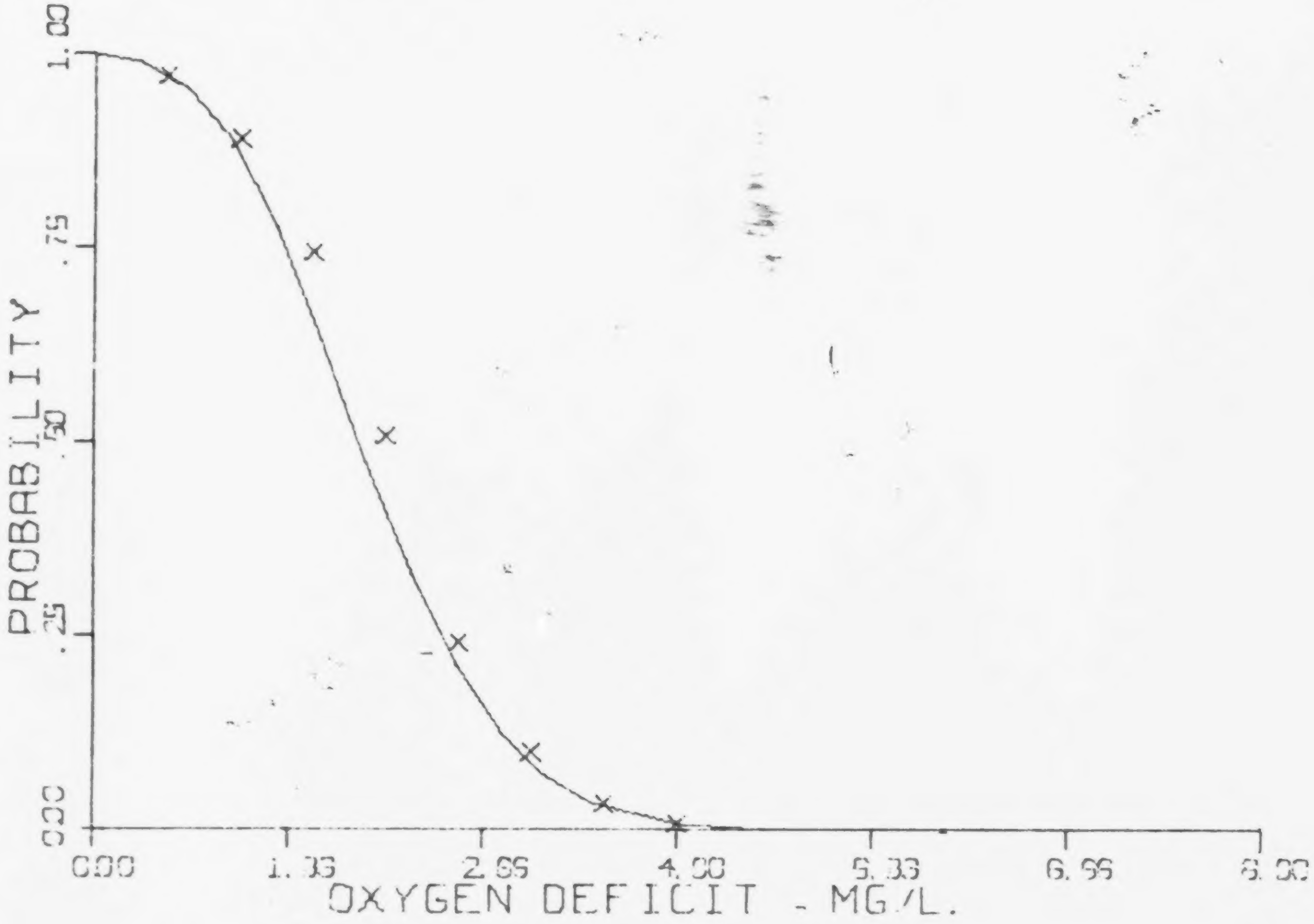
STATION THREE
PREDICTED MEAN OD LEVEL AND OBSERVED HOURLY AVERAGE OD LEVELS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 77



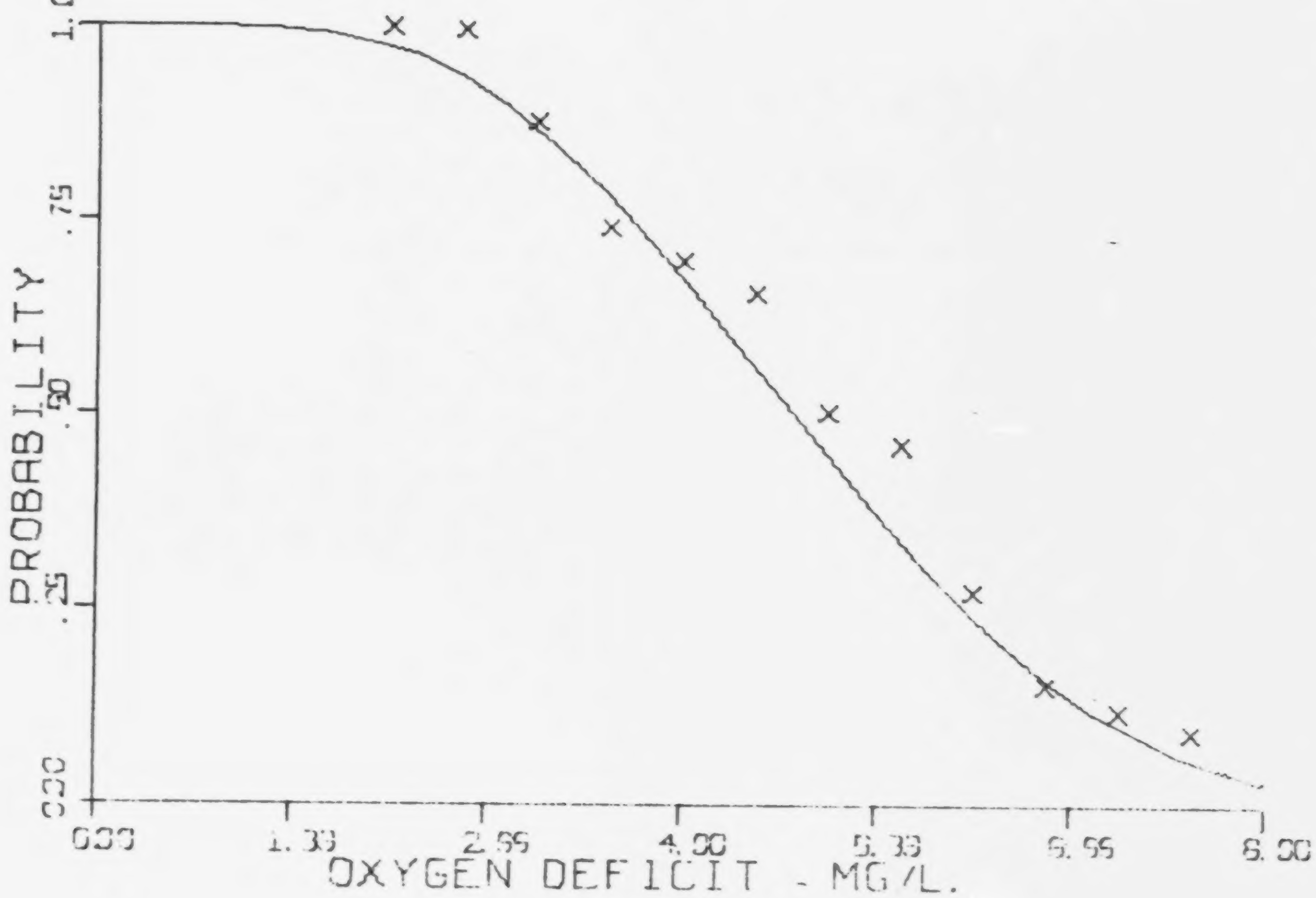
STATION FOUR
PREDICTED MEAN OD LEVEL AND OBSERVED HOURLY AVERAGE OD LEVELS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 79



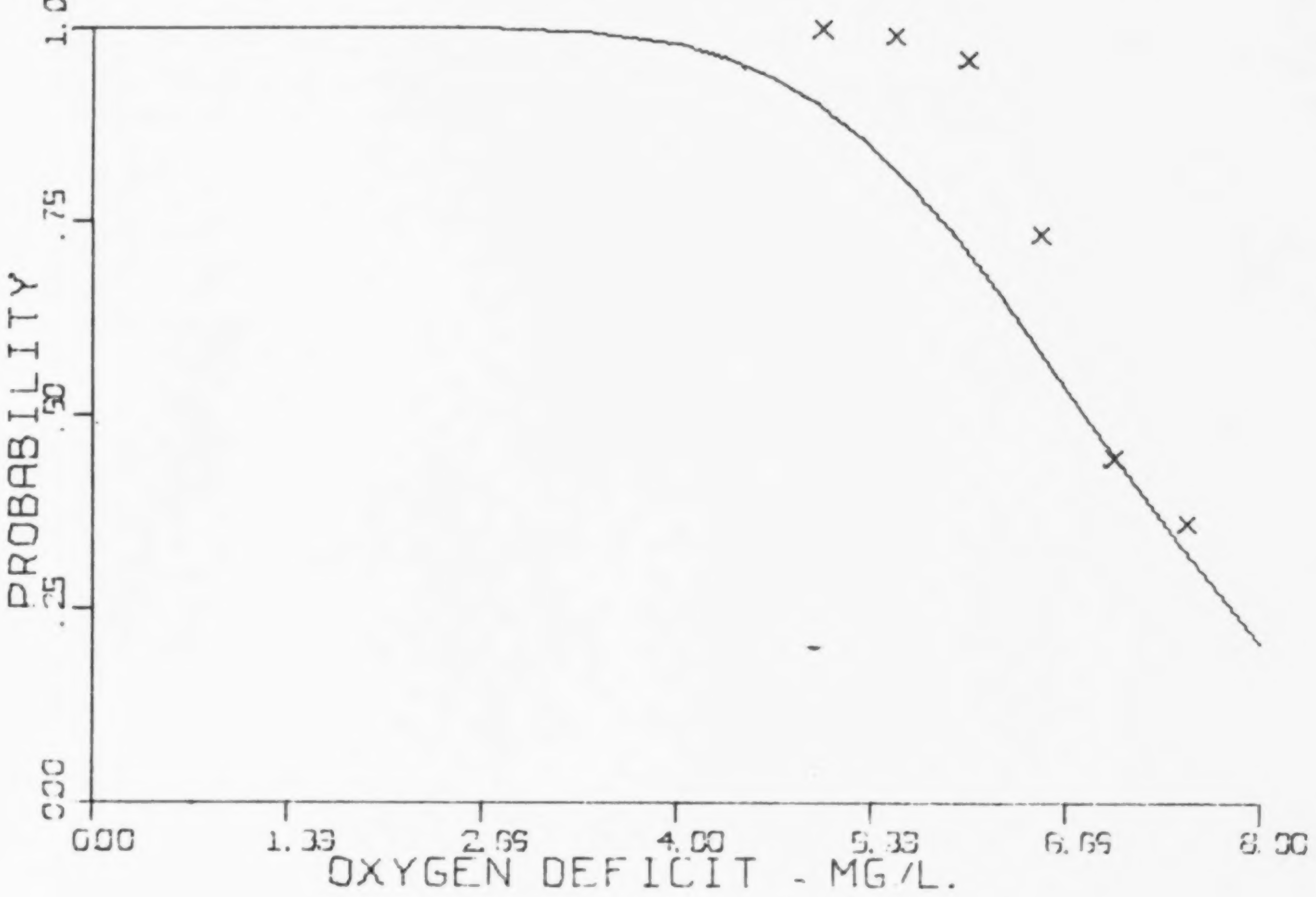
STATION ONE
PREDICTED AND OBSERVED CUMULATIVE OD DISTRIBUTIONS
—— PREDICTED VALUES X OBSERVED VALUES

FIGURE 79



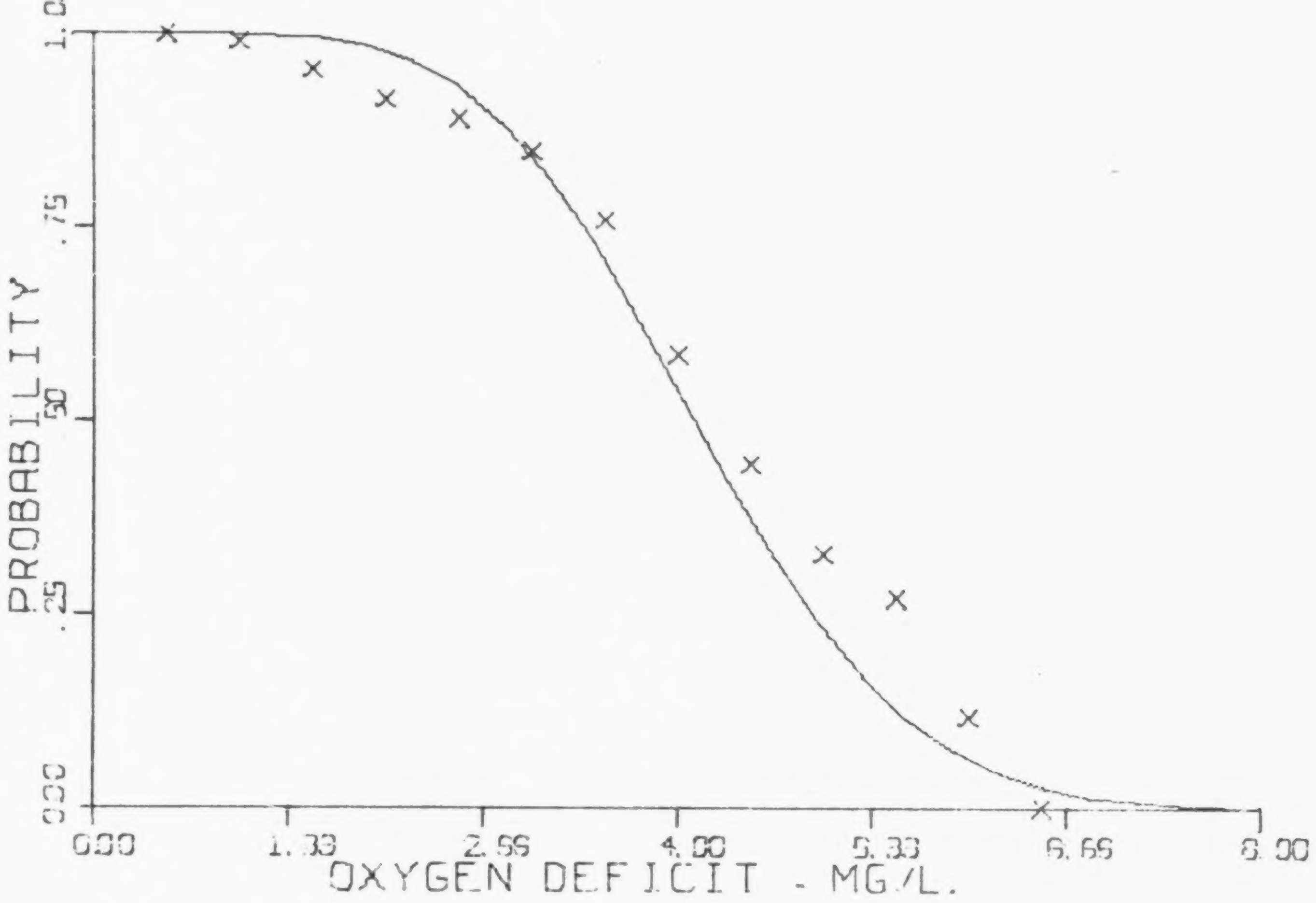
STATION TWO
PREDICTED AND OBSERVED CUMULATIVE OD DISTRIBUTIONS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 60



STATION THREE
PREDICTED AND OBSERVED CUMULATIVE OD DISTRIBUTIONS
—— PREDICTED VALUES X - OBSERVED VALUES

FIGURE 61



STATION FOUR
PREDICTED AND OBSERVED CUMULATIVE OD DISTRIBUTIONS

FIGURE 62

APPENDIX A

PROGRAMS SUPPLEMENTARY TO "NON-DISPERSION MODEL"

PROGRAM 1

```
C     .....................................................
C
C
C     POINT SOURCE
C     CONCENTRATION VS DISTANCE
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10      RATE BOD ENTERS             LBS./HR.
C        FIELD 11-20      CROSS-SECTIONAL AREA        SQ.  FT.
C        FIELD 21-30      FRESH WATER VELOCITY        FT./HR.
C        FIELD 31-40      TIDAL VELOCITY AMP.         FT./HR.
C        FIELD 41-50      TIDAL FREQUENCY             RAD./HR.
C        FIELD 51-60      COE. OF DEOXYGENATION       PER DAY
C        FIELD 61-70      COE. OF REAERATION          PER DAY
C        FIELD 71-80      INITIAL BOD CON.            MG./L.
C      SECOND CARD
C        FIELD  1-10      INITIAL DO CON.             MG./L.
C        FIELD 11-20      ACC. OF DISPLACEMENT EQ. HR.
C      THIRD CARD
C        FIELD  1-10      INITIAL LOCATION            MI.
C        FIELD 11-20      FINAL LOCATION              MI.
C        FIELD 21-30      INC. IN DISTANCE            FT.
C      FOLLOWING CARDS
C        FIELD  1-10      TIME OF INTEREST            HR.
C        AFTER LAST TIME OF INTEREST READ CARD WITH
C        13.0 IN FIELD 1-10.
C
C      PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C
C     SUBROUTINES CALLED
C        DISEQS
C
C
C     .....................................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(1500),P(1500),O(1500),T(100)
C
C
  214 FORMAT(1H1,11HCONT. CURVE//1H )
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16HTIME IN CYCLE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HI =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8HDISTANCE,5X,13H     BOD     ,8X,5X,
     113HOXYGEN DEFICT)
  140 FORMAT(1H ,5X,F7.2,8X,F7.2,17X,F7.2)
  150 FORMAT(14H U= 0 FOR J = ,I3,2X,11HSET U = 100)
  160 FORMAT(5H U = F8.2,2X,8HFOR J = ,I3)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,ALO,DO0,DELTAT
      READ(5,100) X1,XF,DELTAX
```

```
    1 READ(5,100) THETA
      IF(THETA-13.)2,99,2
    2 WRITE(6,110) THETA
      N=0
      WRITE(6,130)
      X=X1*5280.
C
C
C     CALCULATION OF BOD AND DO FOR POINT SOURCE
C
   10 CALL DISEQS (X,T,I)
   11 J=1
      AL3=AL0
      DO3=DO0
   12 U=ABS (U0+U1*COS (OMEGA*(T(J)+THETA)))
      IF(U) 13,13,14
   13 PRINT 150,J
      U=100.0
      GO TO 16
   14 IF(U-U0)15,15,16
   15 PRINT 160,U,J
   16 AL1=AL3+(S/(AREA*U))*16012.0
      TIME=(T(J+1)-T(J))/24.0
      AL2=AL1*EXP (-DK*TIME)
     1-EXP(-TK*TIME)))/(TK-DK)
      DO=DO3*EXP(-TK*TIME)+(DK*AL1*(EXP(-DK*TIME)
      DO3=DO
      AL3=AL2
      IF(J-(I-1))17,18,18
   17 J=J+1
      GO TO 12
   18 IF(I-1) 19,19,3
   19 AL2=0.0
C
C
C     PRINT RESULTS
    3 XN=X/5280.0
      WRITE(6,120) XN,AL2,I,DO
      N=N+1
      R(N)=XN
      P(N)=AL2
      Q(N)=DO
      IF(XN-XF)4,94,94
    4 X=X+DELTAX
      GO TO 10
C     END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END
```

PROGRAM 2

```
      .........................................................
C
C     POINT SOURCE
C     CONCENTRATION VS TIME
C
C     USAGE
C      ALL PARAMETERS REAL
C       FIRST CARD
C         FIELD  1-10      RATE BOD ENTERS            LBS./HR.
C         FIELD 11-20      CROSS-SECTIONAL AREA       SQ.  FT.
C         FIELD 21-30      FRESH WATER VELOCITY       FT./HR.
C         FIELD 31-40      TIDAL VELOCITY AMP.        FT./HR.
C         FIELD 41-50      TIDAL FREQUENCY            RAD./HR.
C         FIELD 51-60      COE. OF DEOXYGENATION      PER DAY
C         FIELD 61-70      COE. OF REAERATION         PER DAY
C         FIELD 71-80      INITIAL BOD CON.           MG./L.
C       SECOND CARD
C         FIELD  1-10      INITIAL DO CON.            MG./L.
C         FIELD 11-20      ACC. OF DISPLACEMENT EQ. HR.
C       THIRD CARD
C         FIELD  1-10      INC. IN TIDAL CYCLE        HR.
C       FOLLOWING CARDS
C         FIELD  1-10      LOCATION OF INTEREST       MI.
C         AFTER LAST LOCATION OF INTEREST READ CARD WITH
C         0.99999 IN FIELD 1-10.
C
C       PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C
C     SUBROUTINES CALLED
C         DISEQS
C
C
C     .........................................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(1500),P(1500),O(1500),T(100)
C
C
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16H      DISTANCE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HI =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8H  TIME  ,5X,13H      BOD     ,8X,5X,
     113HOXYGEN DEFICT)
  140 FORMAT(1H ,5X,F7.2,8X,F7.2,17X,F7.2)
  150 FORMAT(14H U= 0 FOR J = ,I3,2X,11HSET U = 100)
  160 FORMAT(5H U = F8.2,2X,8HFOR J = ,I3)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,ALO,DOO,DELTAT
      READ(5,100) DEL
C
       PRINT 214
C
    1 READ(5,100) X
      IF(X-99999.)2,99,2
    2 WRITE(6,110) X
      N=0
      WRITE(6,130)
      THETA=0.0
      X=5280.0*X
```

```
C      CALCULATION OF BOD AND DO FOR POINT SOURCE
C
   10 CALL DISEOS (X,T,I)
   11 J=1
      AL3=AL0
      DO3=DO0
   12 U=ABS (U0+U1*COS (OMEGA*(T(J)+THETA)))
      IF(U) 13,13,14
   13 PRINT 150,J
      U=100.0
      GO TO 16
   14 IF(U-U0)15,15,16
   15 PRINT 160,U,J
   16 AL1=AL3+(S/(AREA*U))*16012.0
      TIME=(T(J+1)-T(J))/24.0
      AL2=AL1*EXP (-DK*TIME)
      DO=DO3*EXP(-TK*TIME)+(DK*AL1*(EXP(-DK*TIME)
     1-EXP(-TK*TIME)))/(TK-DK)
      DO3=DO
      AL3=AL2
      IF(J-(I-1))17,18,18
   17 J=J+1
      GO TO 12
   18 IF(I-1) 19,19,3
   19 AL2=0.0
C
C
C      PRINT RESULTS
C
    3 WRITE(6,120)THETA,AL2,I,DO
      N=N+1
      R(N) = THETA
      P(N)=AL2
      O(N)=DO
      THETA=THETA+DEL
      IF(THETA-(2.*3.14)/OMEGA)10,10,94
C      END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END
```

PROGRAM 3

```
      ....................................................

C     SEGMENTED SOURCE
C     CONCENTRATION VS DISTANCE
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10      RATE BOD ENTERS           LBS./HR.
C        FIELD 11-20      CROSS-SECTIONAL AREA      SQ.  FT.
C        FIELD 21-30      FRESH WATER VELOCITY      FT./HR.
C        FIELD 31-40      TIDAL VELOCITY AMP.       FT./HR.
C        FIELD 41-50      TIDAL FREQUENCY           RAD./HR.
C        FIELD 51-60      COE. OF DEOXYGENATION     PER DAY
C        FIELD 61-70      COE. OF REAERATION        PER DAY
C        FIELD 71-80      INITIAL BOD CON.          MG./L.
C      SECOND CARD
C        FIELD  1-10      INITIAL DO CON.           MG./L.
C        FIELD 11-20      ACC. OF DISPLACEMENT EQ. HR.
C      THIRD CARD
C        FIELD  1-10      INITIAL LOCATION          MI.
C        FIELD 11-20      FINAL LOCATION            MI.
C        FIELD 21-30      INC. IN DISTANCE          FT.
C      FOLLOWING CARDS
C        FIELD  1-10      TIME OF INTEREST          HR.
C        AFTER LAST TIME OF INTEREST READ CARD WITH
C        13.0 IN FIELD 1-10.
C
C      PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C
C     SUBROUTINES CALLED
C        DISEQS
C        ORDER
C
C
C     ....................................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(600),P(600),D(600),TX(100),TY(100),TZ(200)
C
C
  214 FORMAT(1H1,19HHISTOGRAM,NO MIXING//1H )
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16HTIME IN CYCLE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HJ =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8HDISTANCE,5X,13H     BOD     ,8X,5X,
     113HOXYGEN DEFICT)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,AL0,D00,DELTAT
      READ(5,100) X1,XF,DELTAX
C
C
    1 READ(5,100)THETA
      IF(THETA-13.)2,99,2
    2 WRITE(6,110) THETA
      N=0
      WRITE(6,130)
      X=X1*5280.
```

```
C     CALCULATION OF BOD AND DO FOR SEGMENTED SOURCE
C
C     FIND SOLUTIONS TO DISPLACEMENT EQ. FOR X
      CALL DISEQS(X,TX,INDEX)
C
C     FIND SOLUTIONS TO DISPLACEMENT EQ. FOR X+DELTAX
   10 W=X+DELTAX
      CALL DISEQS (W,TY,I)
C
C     COMBINING BOTH ARRAYS INTO TZ ARRAY
      DO 11 JJ=1,INDEX
   11 TZ(JJ)=TX(JJ)
      IND=INDEX+I
      INDEX=INDEX+1
      DO 12 JJ=INDEX,IND
      JJJ=JJ-INDEX+1
   12 TZ(JJ)=TY(JJJ)
C
C
C     GO TO ORDERING ROUTINE
      CALL ORDER (TZ,IND)
C
      IZ=IND
      IF(IZ-2)14,13,14
   13 AL2=0.0
      DO=0.0
      J=0
      I=1
      TY(I)=0.
      GO TO 3
C
C     CHECK TO SEE IF IZ IS EVEN
   14 IP=IZ
      IP=2*(IP/2)
      IF(IZ-IP)15,16,15
   15 IZ=IZ+1
      TZ(IZ)=0.0
   16 J=1
      AL3=AL0
      DO3=DO0
   17 DT=ABS(TZ(J)-TZ(J+1))
      TIME=(TZ(J+3)+TZ(J+2))/2.-(TZ(J+1)+TZ(J))/2.
      TIME=TIME/24.
      AL1=AL3+S*DT*16012./(AREA*DELTAX)
      AL2=AL1*EXP (-DK*TIME)
      DO=DO3*EXP(-TK*TIME)+(DK*AL1*(EXP(-DK*TIME)
     1-EXP(-TK*TIME)))/(TK-DK)
      DO3=DO
      AL3=AL2
      IF(J-(IZ-3))18,19,99
   18 J=J+2
      GO TO 17
   19 J=(J+1)/2
C
C
C     PRINT RESULTS
    3 XN=X/5280.0
      WRITE(6,120) XN,AL2,J,DO
      N=N+1
      R(N)=XN
      P(N)=AL2
      Q(N)=DO
      IF(XN-XF)4,94,94
```

```
C     PLACE TY'S IN VECTOR TX
    4 DO 5 INDEX=1,I
    5 TX(INDEX) = TY(INDEX)
      INDEX = I
      X=X+DELTAX
      GO TO 10
C     END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END
```

PROGRAM 4

```
      .....................................................

C     SEGMENTED SOURCE
C     CONCENTRATION VS TIME
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10     RATE BOD ENTERS            LBS./HR.
C        FIELD 11-20     CROSS-SECTIONAL AREA       SQ.  FT.
C        FIELD 21-30     FRESH WATER VELOCITY       FT./HR.
C        FIELD 31-40     TIDAL VELOCITY AMP.        FT./HR.
C        FIELD 41-50     TIDAL FREQUENCY            RAD./HR.
C        FIELD 51-60     COE. OF DEOXYGENATION      PER DAY
C        FIELD 61-70     COE. OF REAERATION         PER DAY
C        FIELD 71-80     INITIAL BOD CON.           MG./L.
C      SECOND CARD
C        FIELD  1-10     INITIAL DO CON.            MG./L.
C        FIELD 11-20     ACC. OF DISPLACEMENT EQ. HR.
C      THIRD CARD
C        FIELD  1-10     INC. IN TIDAL CYCLE        HR.
C        FIELD 11-20     INC. IN DISTANCE           FT.
C      FOLLOWING CARDS
C        FIELD  1-10     LOCATION OF INTEREST       MI.
C        AFTER LAST LOCATION OF INTEREST READ CARD WITH
C        0.99999 IN FIELD 1-10.
C
C      PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C
C     SUBROUTINES CALLED
C        DISEQS
C        ORDER
C
C
C     .....................................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(600),P(600),D(600),TX(100),TY(100),TZ(200)
C
C
  214 FORMAT(1H1,19HHISTOGRAM,NO MIXING//1H )
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16H     DISTANCE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HI =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8H  TIME  ,5X,13H     BOD     ,8X,5X,
     113HOXYGEN DEFICT)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,ALO,DO0,DELTAT
      READ(5,100) DEL, DELTAX
C
C
    1 READ(5,100) X
      IF(X-99999.)2,99,2
    2 WRITE(6,110) X
      N=0
      WRITE(6,130)
      THETA=0.0
      X=5280.0*X
```

```
C      CALCULATION OF BOD AND DO FOR SEGMENTED SOURCE
C
C      FIND SOLUTIONS TO DISPLACEMENT EQ. FOR X
   10  CALL DISEQS(X,TX,INDEX)
C
C      FIND SOLUTIONS TO DISPLACEMENT EQ. FOR X+DELTAX
       W=X+DELTAX
       CALL DISEQS (W,TY,I)
C
C      COMBINING BOTH ARRAYS INTO TZ ARRAY
       DO 11 JJ=1,INDEX
   11  TZ(JJ)=TX(JJ)
       IND=INDEX+I
       INDEX=INDEX+1
       DO 12 JJ=INDEX,IND
       JJJ=JJ-INDEX+1
   12  TZ(JJ)=TY(JJJ)
C
C
C      GO TO ORDERING ROUTINE
       CALL ORDER (TZ,IND)
C
       IZ=IND
       IF(IZ-2)14,13,14
   13  AL2=0.0
       DO=0.0
       J=0
       I=1
       TY(I)=0.
       GO TO 3
C
C      CHECK TO SEE IF IZ IS EVEN
   14  IP=IZ
       IP=2*(IP/2)
       IF(IZ-IP)15,16,15
   15  IZ=IZ+1
       TZ(IZ)=0.0
   16  J=1
       AL3=ALO
       DO3=DOO
   17  DT=ABS(TZ(J)-TZ(J+1))
       TIME=(TZ(J+3)+TZ(J+2))/2.-(TZ(J+1)+TZ(J))/2.
       TIME=TIME/24.
       AL1=AL3+S*DT*16012./(AREA*DELTAX)
       AL2=AL1*EXP (-DK*TIME)
       DO=DO3*EXP(-TK*TIME)+(DK*AL1*(EXP(-DK*TIME)
      1-EXP(-TK*TIME)))/(TK-DK)
       DO3=DO
       AL3=AL2
       IF(J-(IZ-3))18,19,99
   18  J=J+2
       GO TO 17
   19  J=(J+1)/2
C
C
C      PRINT RESULTS
```

```
    3 WRITE(6,120)THETA,AL2,J,DO
      N=N+1
      R(N) = THETA
      P(N)=AL2
      O(N)=DO
      THETA=THETA+DEL
      IF(THETA-(2.*3.14)/OMEGA)10,10,94
C     END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END
```

PROGRAM 5

```
      ........................................

C     LINE SOURCE
C     CONCENTRATION VS DISTANCE
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10     RATE BOD ENTERS             LBS
C        FIELD 11-20     CROSS-SECTIONAL AREA        SQ.
C        FIELD 21-30     FRESH WATER VELOCITY        FT.
C        FIELD 31-40     TIDAL VELOCITY AMP.         FT.
C        FIELD 41-50     TIDAL FREQUENCY             RAD
C        FIELD 51-60     COE. OF DEOXYGENATION       PER
C        FIELD 61-70     COE. OF REAERATION          PER
C        FIELD 71-80     INITIAL BOD CON.            MG.
C      SECOND CARD
C        FIELD  1-10     INITIAL DO CON.             MG.
C        FIELD 11-20     ACC. OF DISPLACEMENT EQ. HR.
C      THIRD CARD
C        FIELD  1-10     INITIAL LOCATION            MI.
C        FIELD 11-20     FINAL LOCATION              MI.
C        FIELD 21-30     INC. IN DISTANCE            FT.
C        FIELD 31-40     SOURCE LENGTH               MI.
C      FOLLOWING CARDS
C        FIELD  1-10     TIME OF INTEREST            HR.
C        AFTER LAST TIME OF INTEREST READ CARD WITH
C        13.0 IN FIELD 1-10.
C
C      PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C     SUBROUTINES CALLED
C        DISEQS
C        ORDER
C
C
C     ..............................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(600),P(600),O(600),TX(100),TY(100),
C
C
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16HTIME IN CYCLE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HJ =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8HDISTANCE,5X,13H     BOD     ,8X
     113HOXYGEN DEFICT)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,AL0,D00,DE
      READ(5,100) X1,XF,DELTAX,SL
C
C
      SL=SL*5280.
      RS=(S*16012.)/(AREA*SL)*24.
```

```
    1 READ(5,100)THETA
      IF(THETA-13.)2,99,2
    2 WRITE(6,110) THETA
      N=0
      WRITE(6,130)
      X=X1*5280.
C
C
C     CALCULATION OF BOD AND DO FOR LINE SOURCE
C
C     FIND SOLUTIONS TO DISPLACEMENT EQUATION FOR
C     X + HALF THE SOURCE LENGTH
   10 W=X+SL/2.
      CALL DISEQS(W,TX,INDEX)
C
C     FIND SOLUTIONS TO DISPLACEMENT EQUATION FOR
C     X - HALF THE SOURCE LENGTH
      W=X-SL/2.
      CALL DISEQS (W,TY,I)
C
C     COMBINING BOTH ARRAYS INTO TZ ARRAY
      DO 11 JJ=1,INDEX
   11 TZ(JJ)=TX(JJ)
      IND=INDEX+I
      INDEX=INDEX+1
      DO 12 JJ=INDEX,IND
      JJJ=JJ-INDEX+1
   12 TZ(JJ)=TY(JJJ)
C
C
C     GO TO ORDERING ROUTINE
      CALL ORDER (TZ,IND)
C
      IZ=IND
      IF(IZ-2)14,13,14
   13 AL2=0.0
      DO=0.0
      J=0
      I=1
      TY(I)=0.
      GO TO 3
C     CONTRIBUTION WHILE INSIDE RANGE OF SOURCE
   14 J=1
      AL3=AL0
      DO3=DO0
   15 DT=ABS(TZ(J)-TZ(J+1))
      TIME=DT/24.
      AL2=AL3*EXP(-DK*TIME)+RS*(1.-EXP(-DK*TIME))/DK
      DO=DO3*EXP(-TK*TIME)+ DK*(AL3-RS/DK)*
     1(EXP(-DK*TIME)-EXP(-TK*TIME))
     2/(TK-DK)+DK*RS*(1.-EXP(-TK*TIME))/(TK*DK)
      DO3=DO
      AL3=AL2
      IF(TZ(J+1))16,18,99
C
C     CONTRIBUTION WHILE OUTSIDE RANGE OF SOURCE.
   16 J=J+1
      DT=ABS(TZ(J)-TZ(J+1))
      TIME=DT/24.
      AL2=AL3*EXP (-DK*TIME)
      DO=DO3*EXP(-TK*TIME)+(DK*AL3*(EXP(-DK*TIME)
     1-EXP(-TK*TIME)))/(TK-DK)
      DO3=DO
      AL3=AL2
      IF(TZ(J+1))17,19,99
   17 J=J+1
      GO TO 15
   18 CONTINUE
      J=(J+1)/2
      GO TO 3
   19 J=J/2
C
C
C     PRINT RESULTS
    3 XN=X/5280.0
      WRITE(6,120) XN,AL2,J,DO
      N=N+1
      P(N) = XN
      P(N)=AL2
      O(N)=DO
      IF(XN-XF)4,94,94
C
C     INCREMENT X AND RETURN TO 10
    4 X=X+DELTAX
      GO TO 10
C     END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END
```

PROGRAM 6

```
      .....................................................
C
C     LINE SOURCE
C     CONCENTRATION VS TIME
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10      RATE BOD ENTERS            LBS./HR.
C        FIELD 11-20      CROSS-SECTIONAL AREA       SQ.  FT.
C        FIELD 21-30      FRESH WATER VELOCITY       FT./HR.
C        FIELD 31-40      TIDAL VELOCITY AMP.        FT./HR.
C        FIELD 41-50      TIDAL FREQUENCY            RAD./HR.
C        FIELD 51-60      COE. OF DEOXYGENATION      PER DAY
C        FIELD 61-70      COE. OF REAERATION         PER DAY
C        FIELD 71-80      INITIAL BOD CON.           MG./L.
C      SECOND CARD
C        FIELD  1-10      INITIAL DO CON.            MG./L.
C        FIELD 11-20      ACC. OF DISPLACEMENT EQ.  HR.
C      THIRD CARD
C        FIELD  1-10      INC. IN TIDAL CYCLE        HR.
C        FIELD 11-20      SOURCE LENGTH              MI.
C      FOLLOWING CARDS
C        FIELD  1-10      LOCATION OF INTEREST       MI.
C        AFTER LAST LOCATION OF INTEREST READ CARD WITH
C        0.99999 IN FIELD 1-10.
C
C      PROGRAM LOOPS FOR NEW SET OF PARAMETERS.
C
C     SUBROUTINES CALLED
C        DISEQS
C        ORDER
C
C
C     .....................................................
C
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION R(600),P(600),O(600),TX(100),TY(100),TZ(200)
C
C
  100 FORMAT(8F10.0)
  110 FORMAT(1H1,16H     DISTANCE = ,F7.2)
  120 FORMAT(1H ,5X,F7.2,8X,F7.2,5X,3HI =,I4,5X,F7.2)
  130 FORMAT(1H0,5X,8H  TIME  ,5X,13H     BOD     ,8X,5X,
     113HOXYGEN DEFICT)
C
C
   99 READ(5,100) S,AREA,U0,U1,OMEGA,DK,TK,AL0,DO0,DELTAT
      READ(5,100) DEL,SL
C
C
      SL=SL*5280.
      RS=(S*16012.)/(AREA*SL)*24.
```

```
    1 READ(5,100) X
      IF(X-99999.)2,99,2
    2 WRITE(6,110) X
      N=0
      WRITE(6,130)
      THETA=0.0
      X=5280.0*X
C
C
C     CALCULATION OF BOD AND DO FOR LINE SOURCE
C
C    .FIND SOLUTIONS TO DISPLACEMENT EQUATION FOR
C     X + HALF THE SOURCE LENGTH
   10 W=X+SL/2.
      CALL DISEQS(W,TX,INDEX)
C
C     FIND SOLUTIONS TO DISPLACEMENT EQUATION FOR
C     X - HALF THE SOURCE LENGTH
      W=X-SL/2.
      CALL DISEQS (W,TY,I)
C
C     COMBINING BOTH ARRAYS INTO TZ ARRAY
      DO 11 JJ=1,INDEX
   11 TZ(JJ)=TX(JJ)
      IND=INDEX+I
      INDEX=INDEX+1
      DO 12 JJ=INDEX,IND
      JJJ=JJ-INDEX+1
   12 TZ(JJ)=TY(JJJ)
C
C
C     GO TO ORDERING ROUTINE
      CALL ORDER (TZ,IND)
C
      IZ=IND
      IF(IZ-2)14,13,14
   13 AL2=0.0
      DO=0.0
      J=0
      I=1
      TY(I)=0.
      GO TO 3
C     CONTRIBUTION WHILE INSIDE RANGE OF SOURCE
   14 J=1
      AL3=ALO
      DO3=DOO
   15 DT=ABS(TZ(J)-TZ(J+1))
      TIME=DT/24.
      AL2=AL3*EXP(-DK*TIME)+RS*(1.-EXP(-DK*TIME))/DK
      DO=DO3*EXP(-TK*TIME)+ DK*(AL3-RS/DK)*
     1(EXP(-DK*TIME)-EXP(-TK*TIME))
     2/(TK-DK)+DK*RS*(1.-EXP(-TK*TIME))/(TK*DK)
      DO3=DO
      AL3=AL2
      IF(TZ(J+1))16,18,99
C     CONTRIBUTION WHILE OUTSIDE RANGE OF SOURCE.
   16 J=J+1
      DT=ABS(TZ(J)-TZ(J+1))
      TIME=DT/24.
      AL2=AL3*EXP (-DK*TIME)
      DO=DO3*EXP(-TK*TIME)+(DK*AL3*(EXP(-DK*TIME)
     1-EXP(-TK*TIME)))/(TK-DK)
      DO3=DO
      AL3=AL2
      IF(TZ(J+1))17,19,99
   17 J=J+1
      GO TO 15
   18 CONTINUE
      J=(J+1)/2
      GO TO 3
   19 J=J/2
C
C
C     PRINT RESULTS
C
    3 WRITE(6,120)THETA,AL2,J,DO
      N=N+1
      R(N) = THETA
      P(N)=AL2
      D(N)=DO
      THETA=THETA+DEL
      IF(THETA-(2.*3.14)/OMEGA)10,10,94
C     END OF PRINT
C
   94 CONTINUE
      GO TO 1
C
  199 CONTINUE
      STOP
      END

C     .................................................
C
C     SUBROUTINE DISEQS
C
C     PURPOSE
C        TO COMPUTE THE VECTOR OF SOLUTIONS TO THE
C        DISPLACEMENT EQUATION.
C
C     USAGE
C        CALL DISEQS (X,T,I)
C
C     DESCRIPTION OF PARAMETERS
C        X        - THE LOCATION OF INTEREST.
C        T        - THE RESULTING VECTOR OF SOLUTIONS.
C        I        - THE DIMENSION OF THE VECTOR T.
C
C     METHOD
C        BEGINNING WITH THE POINT WHERE THE LOWER SLOPE LINE
C        INTERSECTS THE X-AXIS THE FUNCTION IS EVALUATED AT
C        EACH MAXIMUM AND MINIMUM POINT.  THE ROOTS ARE
C        OBTAINED BY MEANS OF THE METHOD OF FALSE POSITION.
C
C     .................................................
C
      SUBROUTINE DISEQS (X,T,I)
C
      COMMON U0,U1,OMEGA,THETA,DELTAT
      DIMENSION T(100)
C
C
      Z(AAK,THETA,OMEGA)=-(2.0*AAK-1.0)*(3.1415)/(2.0*OMEGA)
     1 -THETA
      F(X,U0,Z0,U1,OMEGA,THETA)=-(X+U0*Z0+(U1/OMEGA)*
     1(SIN(OMEGA*(Z0+THETA))-SIN(THETA*OMEGA)))
```

```
      A=(U1*(SIN (OMEGA*THETA)-1.0))/(OMEGA*U0)-X/U0
      T(1)=0.0
      T(2)=0.0
C
C
C     LOCATION OF LAST MAX. OR MIN.
C
      K=0
    1 AAK=K
      ZK=Z(AAK,THETA,OMEGA)
      IF(ZK-A)3,2,2
    2 K=K+1
      GO TO 1
C
C     CALCULATION OF ROOTS.
C
C
    3 I=0
      K=K+1
    4 K=K-1
      AAK=K
      ZOLD=1.0
      Z0=Z(AAK,THETA,OMEGA)
      Y0=F(X,U0,Z0,U1,OMEGA,THETA)
      IF(Y0)8,5,8
    5 ZNEW=Z0
    6 I=I+1
      IF(ZNEW)7,20,20
    7 T(I)=ZNEW
      GO TO 4
    8 L=K-1
      AAL=L
      Z1=Z(AAL,THETA,OMEGA)
      Y1=F(X,U0,Z1,U1,OMEGA,THETA)
      IF(Y1)10,9,10
    9 ZNEW=Z1
      GO TO 6
   10 IF(Y0)12,12,11
   11 IF(Y1)14,4,4
   12 IF(Y1)13,4,14
   13 I=I+1
      GO TO 20
   14 ZNEW=(Y1*Z0-Y0*Z1)/(Y1-Y0)
      Y=F(X,U0,ZNEW,U1,OMEGA,THETA)
      IF(ABS(ZNEW-ZOLD)-DELTAT)6,15,15
   15 IF(Y)17,17,16
   16 IF(Y0)18,19,19
   17 IF(Y0)19,19,18
   18 Y1=Y
      Z1=ZNEW
      ZOLD=ZNEW
      GO TO 14
   19 Y0=Y
      Z0=ZNEW
      ZOLD=ZNEW
      GO TO 14
   20 T(I)=0.0
      RETURN
      END
```

```
         ..........................................................
   C     SUBROUTINE ORDER
   C
   C     PURPOSE
   C        TO ORDER THE ELEMENTS OF A VECTOR IN ASCENDING
   C        ORDER.
   C
   C     USAGE
   C        CALL ORDER (TZ,IND)
   C
   C     DESCRIPTION OF PARAMETERS
   C        TZ      - THE VECTOR TO BE ORDERED.
   C        IND     - THE DIMENSION OF THE VECTOR TZ.
   C
   C     METHOD
   C        BEGINNING WITH I=1,T(I) IS COMPARED TO T(I+1),
   C        IF T(I) IS LARGER THE ELEMENTS ARE INTERCHANGED.
   C        THIS IS REPEATED UNTIL NO ADDITIONAL CHANGES IN
   C        THE ORDER ARE REQUIRED.
   C
   C
   C     ..........................................................
   C
         SUBROUTINE ORDER (TZ,IND)
   C
         DIMENSION TZ(1)
   C
   C
         MM=IND-1
         DO 3  JJ=1,MM
         LL=0
         KK=IND-JJ
         DO 2  II=1,KK
         IF(TZ(II)-TZ(II+1))2,2,1
       1 AB=TZ(II)
         TZ(II)=TZ(II+1)
         TZ(II+1)=AB
         LL=1
       2 CONTINUE
         IF(LL)3,4,3
       3 CONTINUE
       4 RETURN
         END
```

APPENDIX B

PROGRAMS SUPPLEMENTARY TO “DIFFUSION MODEL”

PROGRAM 7

```
      ......................................................
C     SOLUTION TO DIFFERENCE EQUATIONS
C     BY ITERATION METHOD
C
C     USAGE
C      ALL PARAMETERS REAL UNLESS SPECIFIED OTHERWISE
C      FIRST CARD
C        FIELD  1-10      FRESH WATER VELOCITY      FT./HR.
C        FIELD 11-20      TIDAL VELOCITY AMP.       FT./HR.
C        FIELD 21-30      TIDAL FREQUENCY           RAD./HR.
C        FIELD 31-40      COE. OF DEOXYGENATION     PER DAY
C        FIELD 41-50      COE. OF REAERATION        PER DAY
C        FIELD 51-60      DIFFUSION COEFFICIENT     SQ.MI./DAY
C        FIELD 61-70      DELTA  T                  DAYS
C      SECOND CARD
C        FIELD  1-10      CUTOFF VALUE ON PROBABILITY
C                         FORMAT E10.1
C        FIELD 11-20      LENGTH OF TIME BETWEEN
C                         PRINT OUT                 DAYS
C        FIELD 21-30      CUTOFF TIME               DAYS
C
C
C     ......................................................
C
C
      DIMENSION U0( 5000),U1( 5000),V0( 5000),V1( 5000)
  100 FORMAT(8F10.0)
  130 FORMAT(1H0,5X,8HDISTANCE,16X,3HBOD,22X,6HDOS-DO)
  140 FORMAT(1H ,5X,F7.2,11X,E16.9,10X,F16.9)
  221 FORMAT(1H1,7HTIME = ,F9.5)
  222 FORMAT(1H ,16X,7HSUMU = ,E16.9,3X,7HSUMV = ,E16.9,
     14X,6HSUM = ,E16.9)
  223 FORMAT(1H ,15X,8HCHECK = ,E16.9,2X,8HCHECK = ,E16.9)
      C(TIME)=VF+VT*(COS(OMEGA*TIME))
C
C     READ PARAMETERS
      READ(5,100)VF,VT,OMEGA,DK,TK,D,DELTAT
      READ (5,101) PROLIM,STEP,TSS
  101 FORMAT (E10.1,2F10.0)
C
C     CALCULATE BASIC TERMS
      DELTAX = SQRT(2.*D*DELTAT)
      TERMA=(1.-DK*DELTAT)/2.
      TERMB=(1.-TK*DELTAT)/2.
      NSTEP = STEP/DELTAT
      R=DK*DELTAT
      VF=VF/220.
      VT=VT/220.
      N=1
      ISENCE = 0
      JSENCE = 0
      NN=2499
      MM=2501
C
C     SET INTIAL CONDITIONS
      U0(2499)=0.
      U0(2500)=1.
      U0(2501)=0.
      V0(2499)=0.
      V0(2500)=0.
      V0(2501)=0.
```

```
C      PRINT, N IS MULT. OF NSTEP
   10  A=N
       TIME=A*DELTAT
       PRINT 221,TIME
C
C      CALCULATE TRANSITION PROBABILITIES
       TERMC=C(TIME)*DELTAT/(DELTAX*2.)
       PA=TERMA+TERMC
       QA=TERMA-TERMC
       PB=TERMB+TERMC
       QB=TERMB-TERMC
       SUMU=0.
       SUMV=0.
       WRITE(6,130)
       U0(NN-1)=0.
       U0(MM+1)=0.
       V0(NN-1)=0.
       V0(MM+1)=0.
C
C      CALCULATE PROBABILITIES FOR T + DELTAT
       DO 11 I=NN,MM
       U1(I)=PA*U0(I-1)+QA*U0(I+1)
       V1(I)=PB*V0(I-1)+QB*V0(I+1)+R*U0(I)
       AI=I
       DIS=(AI-2500.)*DELTAX
       SUMU=SUMU+U1(I)
       SUMV=SUMV+V1(I)
       WRITE(6,140)DIS,U1(I),V1(I)
       IF(U1(I)-PROLIM)21,21,11
   21  IF(V1(I)-PROLIM)22,22,11
   22  IF(DIS)23,23,24
   23  ILOW=I
       ISENCE=1
       GO TO 11
   24  IF(JSENCE)11,25,11
   25  ITOP=I
       JSENCE=1
   11  CONTINUE
C
C      CALCULATE SUMS AND CHECKS
       SUM=SUMU+SUMV
       PRINT 222, SUMU,SUMV,SUM
       CHECKU=EXP(-DK*TIME)
       CHECKV=DK*(EXP(-DK*TIME)-EXP(-TK*TIME))/(TK-DK)
       PRINT 223, CHECKU,CHECKV
       IF(TIME-TSS)12,14,14
C
C      NO PRINT,N IS NOT MULT. OF NSTEP
   50  A=N
       TIME=A*DELTAT
C
C      CALCULATE TRANSITION PROBABILITIES
       TERMC=C(TIME)*DELTAT/(DELTAX*2.)
       PA=TERMA+TERMC
       QA=TERMA-TERMC
       PB=TERMB+TERMC
       QB=TERMB-TERMC
       U0(NN-1)=0.
       U0(MM+1)=0.
       V0(NN-1)=0.
       V0(MM+1)=0.
```

```
C     CALCULATE PROBABILITIES FOR T + DELTAT
      DO 51 I=NN,MM
      U1(I)=PA*UO(I-1)+QA*UO(I+1)
      V1(I)=PB*VO(I-1)+QB*VO(I+1)+R*UO(I)
      IF(U1(I)-PROLIM)61,61,51
   61 IF(V1(I)-PROLIM)62,62,51
   62 IF(I-2500)63,63,64
   63 ILOW=I
      ISENCE=1
      GO TO 51
   64 IF(JSENCE)51,65,51
   65 ITOP=I
      JSENCE=1
   51 CONTINUE
C
C     TRANSPOSE T + DELTAT PROBABILITIES INTO
C     PROBABILITIES FOR T
   12 DO 13 J=NN,MM
      UO(J)=U1(J)
   13 VO(J)=V1(J)
      IF(ISENCE)71,72,71
   71 NN=ILOW-1
      GO TO 73
   72 NN=NN-1
   73 IF(JSENCE)74,75,74
   74 MM=ITOP+1
      GO TO 76
   75 MM=MM+1
   76 ISENCE=0
      JSENCE=0
      N=N+1
C
C     CHECK IF N IS MULT. OF NSTEP
      NT=N/NSTEP
      NT=NT*NSTEP
      IF(N-NT)50,10,50
   14 CONTINUE
      STOP
      END
```

PROGRAM 8

```
      ..................................................
C     SOLUTION TO DIFFERENCE EQUATIONS
C     BY GENERATING FUNCTIONS
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10      FRESH WATER VELOCITY      FT./HR.
C        FIELD 11-20      TIDAL VELOCITY AMP.       FT./HR.
C        FIELD 21-30      TIDAL FREQUENCY           RAD./HR.
C        FIELD 31-40      COE. OF DEOXYGENATION     PER DAY
C        FIELD 41-50      COE. OF REAERATION        PER DAY
C        FIELD 51-60      DIFFUSION COEFFICIENT     SQ.MI./DAY
C        FIELD 61-70      DELTA  T                  DAYS
C      FOLLOWING CARDS
C        FIELD  1-10      TIME OF INTEREST          DAYS
C        PROGRAM STOPS ON END OF DATA
C
C     SUBROUTINES CALLED
C        MULT1
C
C
C     ..................................................
C
C
      DIMENSION CO(1000,2),RESULT(1001),PA(1000),QA(1000),
     1QB(1000),DDO(1000) ,PB(1000)
  100 FORMAT(7F10.0)
  130 FORMAT(1H0,5X,8HDISTANCE,16X,3HBOD,22X,6HDOS-DO)
  140 FORMAT(1H ,5X,F7.2,11X,E16.9,10X,F16.9)
  221 FORMAT(1H1,7HTIME = ,F7.3)
  222 FORMAT(1H ,16X,7HSUMU = ,F16.9,3X,7HSUMV = ,E16.9,
     14X,6HSUM = ,E16.9)
      C(TIME)=VF+VT*(COS(OMEGA*TIME))
C
C     READ PARAMETERS
      READ(5,100)VF,VT,OMEGA,DK,TK,D,DELTAT
      DELTAX = SQRT(2.*D*DELTAT)
C
C     CALCULATE BASIC TERMS
      TERMA=(1.-DK*DELTAT)/2.
      TERMB=(1.-TK*DELTAT)/2.
      R=DK*DELTAT
      VF=VF/220.
      VT=VT/220.
C
C     CALCULATION OF STATE PROBABILITIES
    1 READ(5,100) TSS
      SUMU=0.
      SUMV=0.
      SUM=0.
      N=1
   20 A=N-1
      TIME=A*DELTAT
      IF(TIME-TSS)21,22,22
   21 TERMC=C(TIME)*DELTAT/(DELTAX*2.)
      PA(N)=TERMA+TERMC
      QA(N)=TERMA-TERMC
      PB(N)=TERMB+TERMC
      QB(N)=TERMB-TERMC
      N=N+1
      GO TO 20
C
C     CALCULATIONS OF COEF. IN GENERATING FUNCTION
   22 N=N-1
      NPLUS1=N+1
      DO 1022 K=1,NPLUS1
 1022 DDO(K)=0.
      DO 23 K=2,N
      CO(K,1)=QB(K)
   23 CO(K,2)=PB(K)
      ISUM=0
   24 CO(ISUM+1,1)=1.
      CO(ISUM+1,2)=0.
      IF(ISUM)99,26,25
   25 CO(ISUM,1)=QA(ISUM)
      CO(ISUM,2)=PA(ISUM)
C
C     GO TO BINOMIAL SUBROUTINE
   26 CALL MULT1(CO,N,RESULT)
   27 IF(ISUM-N)28,30,99
   28 DO 29 K=1,NPLUS1
   29 DDO(K)=DDO(K)+RESULT(K)
      ISUM=ISUM+1
      GO TO 24
C     PRINT
   30 PRINT 221,TIME
      PRINT 130
      NSTOP=2*N+1
      DO 33 L=1,NSTOP,2
      KOE=(L+1)/2
      AL=L-N-1
      DIS=AL*DELTAX
      BOD=RESULT(KOE)
      OX=0.
      SUMU=SUMU+BOD
      SUMV=SUMV+OX
      PRINT 140,DIS,BOD,OX
      AL=L-N
      DIS=AL*DELTAX
      BOD=0.
      OX=R*DDO(KOE)
      SUMU=SUMU+BOD
      SUMV=SUMV+OX
   33 PRINT 140,DIS,BOD,OX
      SUM=SUMU+SUMV
      PRINT 222, SUMU,SUMV,SUM
      GO TO 1
   99 STOP
      END
```

PROGRAM 9

```
      ..........................................................
C     COMPLETE BOD AND OD DISTRIBUTIONS
C     BY GENERATING FUNCTIONS
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10      FRESH WATER VELOCITY        FT./HR.
C        FIELD 11-20      TIDAL VELOCITY AMP.         FT./HR.
C        FIELD 21-30      TIDAL FREQUENCY             RAD./HR.
C        FIELD 31-40      COE. OF DEOXYGENATION       PER DAY
C        FIELD 41-50      COE. OF REAERATION          PER DAY
C        FIELD 51-60      DIFFUSION COEFFICIENT       SQ.MI./DAY
C        FIELD 61-70      DELTA  T                    DAYS
C      FOLLOWING CARDS
C        FIELD  1-10      TIME OF INTEREST            DAYS
C        FIELD 11-20      LOCATION OF INTEREST        MI.
C        FIELD 21-30      DELTA  X                    MI.
C        PROGRAM STOPS ON END OF DATA
C
C     SUBROUTINES CALLED
C        MULT2
C
C
C     ..........................................................
C
C
      DIMENSION A(5000,2),B(5000,2)
  100 FORMAT(7F10.0)
  130 FORMAT(1H0,5X,8HREL TIME,16X,3HBOD,22X,6HDOS-DO)
  140 FORMAT(1H ,5X,F9.5,11X,F16.9,10X,E16.9,10X,E16.9,
     110X,E16.9)
  150 FORMAT(1H ,3X,3HBOD)
  160 FORMAT(1H ,3X,5HSTATE,12X,4HPRO.)
  170 FORMAT(1H ,3X,3HDO )
  180 FORMAT(1H ,7HMEAN = ,F9.4,13H  VARIANCE = ,F9.4)
  221 FORMAT(1H1,7HTIME = ,F9.5)
  222 FORMAT(1H ,20HDISTANCE FROM SOURCE,F7.2)
C
C     READ PARAMETERS
      READ(5,100)VF,VT,OMEGA,DK,TK,D,DELTAT
      STEP=2.*DELTAX
      VF=VF/220.
      VT=VT/220.
   10 READ(5,100)TSS,XX,DELTAX
      PRINT 221,TSS
      PRINT 222,XX
      PRINT 130
      TO=0.
      N=0
C
C     CALCULATE SOLUTIONS TO DIFFERENTIAL EQUATIONS
   11 T=TSS-TO
      X1=VF*T+(VT/OMEGA)*(SIN(OMEGA*TSS)-SIN(OMEGA*TO))
      X=XX-X1
      VAR=4.*D*T
      SQRVAR=SQRT( 3.1416*VAR)
      COE=DK/(TK-DK)
      DECAY1=EXP(-DK*T)
      DECAY2=DECAY1-EXP(-TK*T)
      BOD=DECAY1*EXP(-X**2/VAR)/SQRVAR
      DO=COF*EXP(-X**2/VAR)*DECAY2 /SQRVAR
      BOD=BOD*STEP
      DO=DO*STEP
      UNBOD=1.-BOD
      UNDO=1.-DO
      WRITE(6,140)TO,BOD,DO,UNBOD,UNDO
      N=N+1
      A(N,1)=UNBOD
      A(N,2)=BOD
      B(N,1)=UNDO
      B(N,2)=DO
      TO=TO+DELTAT
      IF(TSS-TO)12,11.11
C
C     CALCULATE MEAN AND VARIANCE
   12 ENBOD=0.
      FPPBOD=0.
      ENDO=0.
      FPPDO=0.
      DO 13 K=1,N
      FPPBOD=FPPBOD+A(K,2)**2
      FPPDO=FPPDO+B(K,2)**2
      ENBOD=ENBOD+A(K,2)
   13 ENDO=ENDO+B(K,2)
      VARBOD=ENBOD-FPPBOD
      VARDO=ENDO-FPPDO
      WRITE(6,150)
      WRITE(6,180) ENBOD, VARBOD
      WRITE(6,170)
      WRITE(6,180) ENDO, VARDO
C
C     CALCULATE STATE PRO. FOR BOD
      PRINT 221,TSS
      PRINT 222,XX
      PRINT 150
      PRINT 160
      CALL MULT2(A,N)
C
C     CALCULATE STATE PRO. FOR DO
      PRINT 221, TSS
      PRINT 222,XX
      PRINT 170
      PRINT 160
      CALL MULT2(B,N)
      GO TO 10
   99 CONTINUE
      STOP
      END
C     ......................................................
C
C     SUBROUTINE MULT1
C
C     PURPOSE
C        TO MULTIPLY A VECTOR OF BINOMIALS.
C
C     USAGE
C        CALL MULT1 (CO,NUM,RESULT)
C
C     DESCRIPTION OF PARAMETERS
C        CO      - THE NUM BY 2 VECTOR OF BINOMIALS.
C        NUM     - THE NUMBER OF BINOMIALS.
C        RESULT  - THE VECTOR OF COEFFICIENTS OF THE
C                  RESULTING POLYNOMIAL.
```

```
C     .......................................................
C
      SUBROUTINE MULT1(CO,NUM,RESULT)
      DIMENSION CO(1000,2),RESULT(1001),STOREA(1000),
     1STOREB(1000)
      RESULT(1)=CO(1,1)
      RESULT(2)=CO(1,2)
      DO 13 K=2,NUM
      DO 11 J=1,K
      STOREA(J)= CO(K,1)*RESULT(J)
   11 STOREB(J)= CO(K,2)*RESULT(J)
      RESULT(1)= STOREA(1)
      DO 12 J=2,K
   12 RESULT(J)= STOREA(J)+STOREB(J-1)
   13 RESULT(K+1)= STOREB(K)
      RETURN
      END
C     .......................................................
C
C     SUBROUTINE MULT2
C
C     PURPOSE
C        TO MULTIPLY A VECTOR OF BINOMIALS.
C
C     USAGE
C        CALL MULT2 (CO,NUM)
C
C     DESCRIPTION OF PARAMETERS
C        CO       - THE NUM BY 2 VECTOR OF BINOMIALS.
C        NUM      - THE NUMBER OF BINOMIALS.
C
C
C     .......................................................
C
C
      SUBROUTINE MULT2(CO,NUM)
      DIMENSION CO(5000,2),RESULT(5001),STOREA(5000),
     1STOREB(5000)
  106 FORMAT(1H ,3X,I3,10X,E16.9)
  107 FORMAT(8H0MEAN = F9.4,13H  VARIANCE = F9.4)
      EXK=0.
      EXKS=0.
      LIM=NUM+1
      RESULT(1)=CO(1,1)
      RESULT(2)=CO(1,2)
      DO 13 K=2,NUM
      DO 11 J=1,K
      STOREA(J)= CO(K,1)*RESULT(J)
   11 STOREB(J)= CO(K,2)*RESULT(J)
      RESULT(1)= STOREA(1)
      DO 12 J=2,K
   12 RESULT(J)= STOREA(J)+STOREB(J-1)
   13 RESULT(K+1)= STOREB(K)
      DO 18 K=1,LIM
      KK=K-1
      AK=KK
      EXK=EXK+AK*RESULT(K)
      EXKS=EXKS+RESULT(K)*(AK**2)
   18 WRITE(6,106)KK,RESULT(K)
      VAR=EXKS-EXK**2
      WRITE(6,107)EXK,VAR
      RETURN
      END
```

APPENDIX C

PROGRAMS SUPPLEMENTARY TO "EVALUATION OF MODELS"

PROGRAM 10

```
      .......................................................
C
C     DIFFUSION MODEL
C         MEAN AND VARIANCE OF BOD AND DO.
C         BOD AND DO VS DISTANCE FROM SOURCE FOR FIXED TIME.
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C         FIELD  1-10          RATE BOD ENTERS                  LBS./DAY
C         FIELD 11-20          CROSS-SECTIONAL AREA             SQ.  FT.
C         FIELD 21-30          DELTA                            MG./L.
C         FIELD 31-40          FRESH WATER VELOCITY             MI./DAY
C         FIELD 41-50          TIDAL VELOCITY AMP.              MI./DAY
C         FIELD 51-60          TIDAL FREQUENCY                  RAD./DAY
C         FIELD 61-70          COE. OF DEOXYGENATION            PER DAY
C         FIELD 71-80          COE. OF REAERATION               PER DAY
C      SECOND CARD
C         FIELD  1-10          DIFFUSION COEFFICIENT            SQ.MI./DAY
C      THIRD CARD
C         FIELD  1-10          INITIAL LOCATION                 MI.
C         FIELD 11-20          FINAL LOCATION                   MI.
C         FIELD 21-30          INC. IN DISTANCE                 MI.
C      FOLLOWING CARDS
C         FIELD  1-10          TIME IN TIDAL CYCLE              DAYS
C         PROGRAM STOPS ON END OF DATA.
C
C     SUBROUTINES REQUIRED
C         INTEGA
C         QTFG
C
C
C     .......................................................
C
C
      COMMON VF,VT,OMEGA,DK,TK,D,KEY
      DIMENSION A(1000), B(1000), C(1000)
C
C
    1 READ(5,100) W,AREA,DELTA,VF,VT,OMEGA,DK,TK
      READ(5,100) D
  100 FORMAT(8F10.0)
C
C
C
C     READ XMIN, XMAX, DELTAX IN MI.
      READ(5,100)XMIN,XMAX,DELTAX
C
C     SET UP FOR INTEGRATION
      COE1 = W/(AREA*SQRT(D*3.1417)) * 3.036
      COE2 = COE1*DK/(TK-DK)
      KEY = 0
   10 X = XMIN
      N = 0
C
C     READ TIME IN DAYS
      READ(5,100) TSS
```

```
C
C      PRINT HEADINGS
       WRITE(6,110) TSS
  110 FORMAT(1H1,16HTIME IN CYCLE = ,F9.4)
       WRITE(6,120)
  120 FORMAT(1H0,5X,8HDISTANCE,5X,13H     BOD     ,5X,
     17H DOS-DO,10X,7HBOD VAR,10X,6HDO VAR)
C
C      INTEGRATION
   11 CALL INTEGA(X,TSS,BOD,DO)
C
C      CALCULATE MEAN AND VARIANCE
       BOD = COE1*BOD
       DO = COE2*DO
       VARBOD = BOD*DELTA
       VARDO = DO*DELTA
C
C      PRINT RESULTS
       WRITE(6,130)X,BOD,DO,VARBOD,VARDO
  130 FORMAT(1H ,5X,F7.2,9X,F7.2,7X,F7.2,10X,F7.2,9X,F7.2)
       N = N+1
       A(N) = X
       B(N) = BOD
       C(N) = DO
       IF(X-XMAX+DELTAX/2.) 12,13,13
C
   12 X = X + DELTAX
       GO TO 11
   13 CONTINUE
       GO TO 10
C
   99 CONTINUE
C
C
       STOP
       END
```

PROGRAM 11

```
C     .................................................
C
C     DIFFUSION MODEL
C        MEAN AND VARIANCE OF BOD AND DO.
C        BOD AND DO VS TIME FOR FIXED DISTANCE FROM SOURCE.
C
C     USAGE
C      ALL PARAMETERS REAL
C      FIRST CARD
C        FIELD  1-10     RATE BOD ENTERS          LBS./DAY
C        FIELD 11-20     CROSS-SECTIONAL AREA     SQ.  FT.
C        FIELD 21-30     DELTA                    MG./L.
C        FIELD 31-40     FRESH WATER VELOCITY     MI./DAY
C        FIELD 41-50     TIDAL VELOCITY AMP.      MI./DAY
C        FIELD 51-60     TIDAL FREQUENCY          RAD./DAY
C        FIELD 61-70     COE. OF DEOXYGENATION    PER DAY
C        FIELD 71-80     COE. OF REAERATION       PER DAY
C      SECOND CARD
C        FIELD  1-10     DIFFUSION COEFFICIENT    SQ.MI./DAY
C      THIRD CARD
C        FIELD  1-10     DISTANCE FROM SOURCE     MI.
C        FIELD 11-20     INC. IN TIME             DAYS
C      FOURTH CARD
C        NUMBER OF CRITICAL BOD LEVELS.
C              FORMAT (I2), MAXIMUM NUMBER IS 6.
C      FIFTH CARD
C        CRITICAL BOD LEVELS.
C              FORMAT(6F10.0), LEVELS MUST BE IN
C              ASCENDING ORDER.
C      SIXTH CARD
C        NUMBER OF CRITICAL  OD LEVELS.
C              FORMAT (I2), MAXIMUM NUMBER IS 6.
C      SEVENTH CARD
C        CRITICAL  OD LEVELS.
C              FORMAT(6F10.0), LEVELS MUST BE IN
C              ASCENDING ORDER.
C      PROGRAM LOOPS BACK TO THIRD CARD FOR NEW LOCATION
C      AND CRITICAL LEVELS.
C        PROGRAM STOPS ON END OF DATA.
C
C     SUBROUTINES REQUIRED
C        INTEGA
C        QTFG
C        POSDIS
C
C
C     .................................................
C
C
      COMMON VF,VT,OMEGA,DK,TK,D,KEY
      DIMENSION A(1000), B(1000), C(1000)
C
C
1     READ(5,100) W,AREA,DELTA,VF,VT,OMEGA,DK,TK
      READ(5,100) D
  100 FORMAT(8F10.0)
```

```
C      SET UP FOR INTEGRATION
       COE1 = W/(AREA*SQRT(D*3.1417)) * 3.036
       COE2 = COE1*DK/(TK-DK)
       KEY = 0
C
   10 READ(5,100) X,DELTAT
       N = 0
       TSS = 0.0
       TF = 6.2832/OMEGA - DELTAT/2.
C
C      PRINT HEADINGS
       WRITE (6,110) X
  110 FORMAT (1H1,23HDISTANCE FROM SOURCE = ,F7.2)
       WRITE (6,120)
  120 FORMAT(1H0,5X,8H  TIME  ,5X,13H     BOD     ,5X,
     17H DOS-DO,10X,7HBOD VAR,10X,6HDO VAR)
C
C      INTEGRATION
   11 CALL INTEGA(X,TSS,BOD,DO)
C
C      CALCULATE MEAN AND VARIANCE
       BOD = COE1*BOD
       DO = COE2*DO
       VARBOD = BOD*DELTA
       VARDO = DO*DELTA
C
C      PRINT RESULTS
       WRITE (6,130) TSS,BOD,DO,VARBOD,VARDO
  130 FORMAT(1H ,5X,F7.4,9X,F7.2,7X,F7.2,10X,F7.2,9X,F7.2)
       N = N+1
       A(N) = TSS * 24.
       B(N) = BOD
       C(N) = DO
       IF(TSS-TF)12,13,13
C
   12 TSS = TSS + DELTAT
       GO TO 11
C
   13 CONTINUE
       TOT = 0.
       DO 20 II=1,N
   20 TOT = TOT + C(II)
       AN = N
       AMEAN = TOT/AN
       WRITE(6,500) AMEAN
  500 FORMAT(1H 13HAVERAGE DO = ,F7.2)
       CALL POSDIS  (N,A,B,DELTA,1)
       CALL POSDIS  (N,A,C,DELTA,2)
       GO TO 10
C
   99 CONTINUE
       STOP
       END

C      ...................................................
C
C      SUBROUTINE INTEGA
C
C      PURPOSE
C         TO COMPUTE THE INTEGRAL FOR THE BOD AND DO FUNCTIONS;
C         GIVEN THE TIME AND LOCATION.
```

```
C     USAGE
C        CALL INTEGA(XX,TSS,BOD,DO)
C
C     DISCREPTION OF PARAMETERS
C        XX      - THE LOCATION OF INTEREST.
C        TSS     - THE TIME OF INTEREST.
C        BOD     - THE RESULTING INTEGRAL VALUE FOR THE
C                  BOD FUNCTION.
C        DO      - THE RESULTING INTEGRAL VALUE FOR THE
C                  DO  FUNCTION.
C
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        QTFG
C
C     METHOD
C        A VECTOR OF T'S IS FORMED AS FOLLOWS:
C        T(1) = -(LOG(1-(1-EXP(-DK*TSS))/N)/DK
C        T(K) = -(LOG(EXP(-DK*T(K-1))-(1-EXP(-DK*TSS))/N)/DK
C        WHERE LOG IS THE NATURAL LOG FUNCTION AND N IS THE
C        NUMBER OF SEGMENTS USED.  THE FUNCTION IS EVALUATED
C        AT EACH VALUE OF T,AND THE INTEGRATION IS PREFORMED
C        BY MEANS OF THE TRAPEZODIAL RULE.
C
C     ...................................................
      SUBROUTINE INTEGA(XX,TSS,BOD,DO)
C
      COMMON VF,VT,OMEGA,DK,TK,D,KEY
      DIMENSION YBOD(500),YDO(500),TT(500)
C
C
      IF(KEY.GE. 1) GO TO 11
      NDIM = 100
      B = 1./FLOAT(NDIM)
      TT(1) = -(ALOG(1.-B))/DK
      DO 10 K=2,NDIM
   10 TT(K) = -(ALOG(EXP(-DK*TT(K-1))-B))/DK
   11 DO 12 K=1,NDIM
      T = TT(K)
      TO = TSS - T
      X1=VF*T+(VT/OMEGA)*(SIN(OMEGA*TSS)-SIN(OMEGA*TO))
      V=(XX-X1)**2/(4.*D*T)
      YBOD(K)=EXP(-V-DK*T)/SQRT(T)
   12 YDO(K)=(EXP(-V-DK*T)-EXP(-V-TK*T))/SQRT(T)
C
C     INTEGRATION
      CALL QTFG(TT,YBOD,YBOD,NDIM)
      CALL QTFG(TT,YDO,YDO,NDIM)
C
      BOD = YBOD(NDIM)
      DO = YDO(NDIM)
      KEY = 1
      RETURN
      END
C     ...................................................
C
C     SUBROUTINE QTFG
C
C     PURPOSE
C        TO COMPUTE THE VECTOR OF INTEGRAL VALUES FOR A GIVEN
C        TABLE OF ARGUMENT VALUES AND FUNCTION VALUES.
C
C     USAGE
C        CALL QTFG (X,Y,Z,NDIM)
```

```
C     DESCRIPTION OF PARAMETERS
C        X       - THE INPUT VECTOR OF ARGUMENT VALUES.
C        Y       - THE INPUT VECTOR OF FUNCTION VALUES.
C        Z       - THE RESULTING VECTOR OF INTEGRAL VALUES.
C                  Z MAY BE IDENTICAL WITH X OR Y.
C        NDIM  . - THE DIMENSION OF VECTORS X,Y,Z.
C
C     METHOD
C        BEGINNING WITH Z(1) = 0, EVALUCATION OF VECTOR Z
C        IS DONE BY MEANS OF THE TRAPEZODIAL RULE (SECOND
C        ORDER FORMULA).
C
C     ..................................................................
C
      SUBROUTINE QTFG(X,Y,Z,NDIM)
C
      DIMENSION X(1), Y(1), Z(1)
C
      SUM2=0.
C
C     INTEGRATION LOOP
    1 DO 2 I=2,NDIM
      SUM1 = SUM2
      SUM2 = SUM2 + .5*(X(I)-X(I-1))*(Y(I)+Y(I-1))
    2 Z(I-1) = SUM1
    3 Z(NDIM) = SUM2
      RETURN
      END




C
C
C     ..................................................................
C
C
C     SUBROUTINE POSDIS
C
C     PURPOSE
C        TO CALCULATE THE EXPECTED PERCENTAGE OF TIME
C        THAT THE BOD OR OD WILL BE ABOVE CRITICAL LEVELS.
C
C     USAGE
C        CALL POSDIS (N,TIME,OD,DELTA,KEY)
C
C     DESCRIPTION OF PARAMETERS
C        N       - THE DIMENSION OF THE VECTORS TIME
C                  AND OD.
C        TIME    - THE VECTORS OF TIMES CORRESPONDING TO
C                  THE INC. IN TIME USED IN MAIN.
C        OD      - THE VECTOR OF MEANS CALCULATED IN MAIN.
C        DELTA   - DELTA
C        KEY     - KEY = 1 INDICATES BOD DISTRIBUTION.
C                  KEY = 2 INDICATES  OD DISTRIBUTION.
C
C      THE FOLLOWING SET OF CARDS MUST BE READ
C      INTO THIS SUBROUTINE:
C        THE NUMBER OF CRITICAL LEVELS; THE MAXMUM NO. IS 6.
C               FORMAT (I2)
C        THE CRITICAL LEVELS; MUST BE IN ASCENDING ORDER.
C               FORMAT (6F10.0)
```

```
C      ...................................................
C
C
C
       SUBROUTINE POSDIS  (N,TIME,OD,DELTA,KEY)
C
       DIMENSION P(5000),TIME(1),OD(1),ODC(10),STATE(10),
      1PRO(10),AVE(10)
       REAL MEAN,LOGFAC,LOGP
       INTEGER STATE
C
C      READ CRITICAL VATUES AND CONVERT TO STATES
       READ(5,100) NUM
   100 FORMAT (I2)
       READ (5,101) (ODC(L),L=1,NUM)
   101 FORMAT (6F10.0)
       DO 1 L=1,NUM
     1 STATE(L)=ODC(L)/DELTA
C
C      PRINT HEADING
       IF(KEY.EQ.1) GO TO 5
       WRITE(6,102) (ODC(L),L=1,NUM)
   102 FORMAT(1H1,6H  TIME,4X,14HOXYGEN DEFICIT,4X,
      116HPRO. OD IS ABOVE,6(4X,F6.2))
       GO TO 6
     5 WRITE(6,103) (ODC(L),L=1,NUM)
   103 FORMAT(1H1,6H  TIME,4X,14H    B.O.D.    ,4X,
      116HPRO.BOD IS ABOVE,6(4X,F6.2))
     6 CONTINUE
C
C      SET UP FOR POISSON
       CUTOFF = .0001
       DO 2 L=1,NUM
     2 AVE(L) = 0.0
       DO 3 L=1,5000
     3 P(L) = 0.0
       DO 30 NN=1,N
       MEAN = OD(NN)/DELTA
C
C      CALCULATION OF STATE PROBABILITIES
       IMEAN=MEAN+2.
C      CALCULATE P(IMEAN)
       LOGFAC=0.
       DO 11 K=2,IMEAN
       AK=K
    11 LOGFAC=LOGFAC + ALOG(AK)
       AIMEAN=IMEAN
       LOGP = -MEAN + AIMEAN*ALOG(MEAN) - LOGFAC
       P(IMEAN) = EXP(LOGP)
C      CALCULATION OF P'S LESS THAN IMEAN
       ICHECK=1
       IMAX = IMEAN-1
       DO 12 K=1,IMAX
       J=IMEAN - K
       AJP1 = J+1
       P(J) = (AJP1/MEAN)* P(J+1)
       IF(P(J)-CUTOFF)13,13,12
    12 CONTINUE
       PO = P(1)/MEAN
       J=1
       ICHECK = 0
C      CALCULATIONS OF P'S GREATER THAN IMEAN
    13 IMIN=IMEAN+1
       DO 14 K=IMIN, 5000
       AK=K
      P(K) = (MEAN/AK)*P(K-1)
      IF(P(K)-CUTOFF)15,15,14
   14 CONTINUE
C
C     CHECK TO SEE IF CRITICAL STATES ARE GREATER THAN K
   15 DO 16 L=1,NUM
      NEWNU = NUM-L+1
      IF(STATE(NEWNU).LT.K) GO TO 20
   16 PRO(NEWNU) = 0.0
C
C     CALCULATE CUMULATIVE PROBABILITIES
   20 SUM = 0.
      IF(ICHECK)22,21,22
   21 SUM = PO
   22 LAST = STATE(1)
      DO 23 L=J,LAST
   23 SUM = SUM + P(L)
      PRO(1) = 1.-SUM
C
      MSTOP = NEWNU-1
      DO 25 M=1,MSTOP
      LFIR = STATE(M)+1
      LAST = STATE(M+1)
      DO 24 L=LFIR,LAST
   24 SUM = SUM+P(L)
      PRO(M+1) = 1.-SUM
   25 CONTINUE
C
C     PRINT RESULTS
      WRITE(6,104) TIME(NN),OD(NN),(PRO(L),L=1,NUM)
  104 FORMAT(1H F7.4,6X,F7.3,24X,6(4X,F6.2))
C
C     FIND AVERAGE
      DO 26 L=1,NUM
   26 AVE(L) = AVE(L)+PRO(L)
   30 CONTINUE
C
C     PRINT AVERAGE PRO.
      AN = N
      DO 31 L=1,NUM
   31 AVE(L) = AVE(L)/AN  * 100.
      IF(KEY.EQ.1) GO TO 7
      WRITE(6,105) (AVE(L), L=1,NUM)
  105 FORMAT(1H0,45HEXPECTED PERCENT OF TIME OD ABOVE GIVEN VALUE,
     16(4X,F6.2))
      GO TO 8
    7 WRITE(6,106) (AVE(L), L=1,NUM)
  106 FORMAT(1H0,46HEXPECTED PERCENT OF TIME BOD ABOVE GIVEN VALUE
     16(4X,F6.2))
    8 CONTINUE
      RETURN
      END


EXECUTION TERMINATING DUE TO ERROR COUNT FOR ERROR NUMBER  217

FIOCS - END OF DATA SET ON UNIT    5
```

```
K FOLLOWS-  ROUTINE   ISN   REG.  14  REG.  15
            IBCOM           0000D380  0000D400
            MAIN            000044C0  7000D020

 REG.   0  REG.   1
 00000049  0000D48C
 00000030  0003FF1C

 NT=  7000D020
OF ERRORS FOR THIS JOB  ERROR NUMBER  NUMBER OF ERRORS
                         217             1
```

BIBLIOGRAPHY

1. Adke, S. R., A Birth, Death, and Diffusion Process, J. Math. Analysis and Application, Vol.7, (1963), pp. 209-224.

2. Adke, S. R., A Stochastic Population Diffusing on a Finite Interval, J. of Indian Statistical Association, Vol. 2, (1964), pp. 32-40.

3. Adke, S. R., The Generalized Birth and Death and Gaussian Diffusion, J. Math. Analysis and Application, Vol. 9, (1964), pp. 336-340.

4. Anderson, Decima M., *Computer Programming Fortran IV,* Appleton-Century-Crofts, New York, 1966.

5. Arrons, A. B. and H. Stommel, A Mixing-Length Theory of Tidal Flushing, Transactions, American Geophysical Union, Vol. 32, (1951), p. 419.

6. Bailey, Norman T. J., *The Elements of Stochastic Processes,* John Wiley and Sons, Inc., New York, 1964.

7. Bharucha-Reid, A. T., *Elements of the Theory of Markov Processes and Their Applications*, McGraw-Hill Book Company, Inc., New York, 1960.

8. Diachishin, Alex N., Waste Disposal in Tidal Waters, ASCE J. Sanitary Eng. Div., Vol. 89, No. SA4, August 1963, pp. 23-43.

9. Di Luzio, Frank C., Federal Government and Estuarine Pollution Abatement, ASCE J. Sanitary Eng. Div., Vol. 94, No. SA2, April 1968, pp. 201-211.

10. Dobbins, W. E., BOD and Oxygen Relationships in Streams, ASCE J. Sanitary Eng. Div., Vol. 90, No. SA3, June 1964, paper 3949, pp. 53-78.

11. Dobbins, W. E., Diffusion and Mixing, Boston Soc. Engs-J., Vol. 52, No. 2, April 1965, pp. 108-128.

12. Doob, J. L., The Brownian Movement and Stochastic Equations, Annals of Math., Vol. 43, (1942), pp. 351-369.

13. Fair, Gordon M. and John C. Geyer, Water and Waste-Water Disposal, John Wiley and Sons, Inc., New York, 1954.

14. Feller, W., An Introduction to Probability Theory and Its Applications, Vol. 1 and 2, John Wiley and Sons, Inc., New York, 1966.

15. Feller, W., Diffusion Processes in Genetics, Proc. of the Second Berkeley Symposium on Math. Stat. and Probability, J. Neyman (ed.) (1950), p. 227.

16. Feller, W., Diffusion Processes in One-Dimension, Transactions of the American Mathematical Society, Vol. 77, No. 1, July 1954, pp. 1-31.

17. Feller, W., Some Recent Trends in the Mathematical Theory of Diffusion, Proc. International Congress of Mathematicians, Vol. 2 (1950), pp. 322-339.

18. Feller, W., Two Singular Diffusion Problems, Annals of Math., Vol. 54, (1951), pp. 173-182.

19. Feller, W., Zur Theorie der Stochastischen Prozesse, Mathematische Annalen, Vol. 113 (1937), pp. 113-160.

20. Haight, Frank A., Handbook of the Poisson Distribution, John Wiley and Sons, Inc., New York, 1967.

21. Harleman, D. R. F., Significance of Longitudinal Dispersion in Analysis of Pollution in Estuaries, Water Pollution Control Federation-J., Vol. 36, No. 3, March 1964, p. 285.

22. Harleman, D. R. F., The Significance of Longitudinal Dispersion in the Analysis of Pollution in Estuaries, Advances in Water Pollution Research Proc. of the Second International Conference held in Tokyo 1964, Pergamon Press, London 1965, pp. 279-306.

23. Harleman, D. R. F., Edward R. Holley, Jr. and Wayne C. Huber, Interpretation of Water Pollution Data from Tidal Estuary Models, Third International Conference on Water Pollution Research, Paper No. 3 (1966).

24. Hetling, L. J., A Mathematical Model for the Potomac River--What It Has Done and What It Can Do, CB-SRBP, Tech. Paper No. 8, FWPCA.

25. Hetling, L. J., and N. A. Jaworski, The Role of Mathematical Models in the Potomac River Basin Water Quality Management Program, Chesapeake Field Station, CB-SRBP, FWPCA, Dec. 1967.

26. Hetling, L. J., and R. L. O'Connell, A Study of Tidal Dispersion in the Potomac River, CB-SRBP Technical Paper No. 7, FWPCA.

27. Hetling, L. J., and R. L. O'Connell, A Study of Tidal Dispersion in the Potomac River, Water Resources Research, Vol. 2, No. 4, 4th Quarter 1966, pp. 825-841.

28. Hetling, L. J., and R. L. O'Connell, An O^3 Balance for the Potomac Estuary, Chesapeake Field Station, Annapolis, Maryland, 1967.

29. Hetling, L. J., and R. L. O'Connell, Estimating Diffusion Characteristics of Tidal Waters, CB-SRBP Technical Paper No. 4, FWPCA.

30. Hetling, L. J., and R. L. O'Connell, Estimating Diffusion Characteristics of Tidal Waters, Water and Sewage Works, Vol. 112, No. 10, October 1965, pp. 378-380.

31. Hildebrand, F. B., Introduction to Numerical Analysis, McGraw-Hill Book Company, New York, 1956.

32. Kac, M., Random Walk and Theory of Brownian Motion, American Math. Monthly, Vol. 54 (1947), pp. 369-391.

33. Karlin, Samuel, A First Course in Stochastic Processes, Academic Press, New York, 1968.

34. Kelchumn, V. H., The Flushing of Tidal Estuaries, Sewage and Industrial Waste, Vol. 23, No. 2, February 1951, p. 198.

35. Kells, Lyman M., Elementary Differential Equations, McGraw-Hill Book Co., New York, 1965.

36. Kent, R., Diffusion in Sectionally Homogeneous Estuary, ASCE J. Sanitary Eng. Div., Vol. 86, No. SA2 March 1960, Paper 2408, pp. 15-47.

37. Kimura, Motoo, Stochastic Processes and Distribution of Gene Frequencies Under Natural Selection, Cold Spring Harbor Symposia on Quantitative Biology, Vol. XX, The Biological Lab. Cold Springs Harbor, L. I., New York, 1955.

38. Klein, Louis, River Pollution II. Causes and Effects, Butterworths, London, 1962.

39. Koopman, B. O., Necessary and Sufficient Conditions for Poisson's Distributions. Proc. American Mathematical Society, December 1950, pp. 813-823.

40. Kothandaraman, Verrasamy, Probabilistic Analysis of Wastewater Treatment and Disposal Systems, Dept. of Civil Engineering, University of Illinois, Urbana, Water Resources Center, Research Report No. 14, June 1968.

41. Lauff, George H., (ed.) Estuaries, Publication No. 83, American Association for the Advancement of Science, Washington, D. C., 1967.

42. Leeds, J. V., and H. H. Bybee, Solution of Estuary Problems and Network Programs, ASCE J. Sanitary Eng. Div., Vol. 93, No. SA3, 1967, pp. 29-36.

43. Loucks, Daniel P., A Probabilistic Analysis of Waste Water Treatment Systems, Cornell University Water Resources Center, September 1965.

44. Lynch, William O., Discussion of "BOD and DO Relationships in Streams" (Dobbins), ASCE J. Sanitary Eng. Div., Vol. 91, No. SA1, February 1965, pp. 82-83.

45. O'Connell, R. L., and C. M. Walter, Hydraulic Model Test of Estuarial Water Dispersion, ASCE J. Sanitary Eng. Div., Vol. 89, No. SA1, January 1963, Paper 3394, pp. 51-65.

46. O'Connor, D. J., Estuarine Distribution of Non-Conservative Substances, ASCE J. Sanitary Eng. Div., Vol. 91, No. SA1, February 1965, Paper 4225, pp. 23-42.

47. O'Connor, D. J., Notes from Summer Institute in Water Pollution Control Stream and Estuarine Analysis, Manhattan College, New York, 1968.

48. O'Connor, D. J., Oxygen Balance of Estuary, ASCE J. Sanitary Eng. Div., Vol. 86, No. SA3, May 1960, Paper 2472, pp. 35-55.

49. O'Connor, D. J., and W. E. Dobbins, The Mechanism of Reaeration in Natural Streams, ASCE J. Sanitary Eng. Div., Vol. 82, No. SA6, December 1956, paper 1115.

50. Prabhu, N. U., Stochastic Processes, The MacMillan Co., New York, 1965.

51. Pritchard, D. W., Estuarine Hydrography, Advances in Geo-Physics, Academic Press, New York, 1952.

52. Pyatt, E. E., On Determining Pollutant Distribution in Tidal Estuary, US Geol. Survey-Water Supply, Paper 1586f, (1964).

53. Rosenblatt, Murray, Random Processes, Oxford University Press, New York, 1962.

54. Sneddon, Ian N., Elements of Partial Differential Equations, McGraw-Hill Book Co., New York, 1957.

55. Stommel, H., Computations on Pollution in a Vertically Mixed Estuary, Sewage and Industrial Waste, Vol. 25, No. 9, September 1953.

56. Streeter, H. W., and E. B. Phelps, A Study of the Pollution and Natural Purification of the Ohio River, Public Health Bulletin 146, USPHS, Washington, D. C. (1925).

57. Taylor, G. I., The Present Position in the Theory of Turbulent Diffusion, Advances in Geophysics, Academic Press, New York, Vol. 6, 1959, pp. 102-110.

58. Thackston, Edward L., and Peter A. Krenkel, Longitudinal Mixing and Reaeration in Natural Streams, Tech. Report No. 7, Sanitary and Water Resources Eng., Dept. of Civil Eng., Vanderbilt University, 1966.

59. Thayer, Richard P., and Richard G. Krutchkoff, A Stochastic Model for Pollution and Dissolved Oxygen in Streams, Water Resources Research Center, Virginia Polytechnic Institute, Blacksburg, Virginia, August 1966.

60. Thayer, Richard P., and Richard G. Krutchkoff, Stochastic Model for BOD and DO in Streams, ASCE J. Sanitary Eng. Div., No. SA3, June 1967, pp. 59-72.

61. Thomann, Robert V., Mathematical Model for Dissolved Oxygen, ASCE J. Sanitary Eng. Div., Vol. 89, No. SA5, October 1963, Paper 3680, p. 30.

62. Thomann, Robert V., Recent Results for Mathematical Model of Water-Pollution Control in Delaware Estuary, Water Resources Research, Vol. 1, No. 3, 3rd Quarter, (1965), pp. 349-359.

63. Thomann, Robert V., Time-Series Analysis of Water-Quality Data, ASCE J. Sanitary Eng. Div., No. SA1, February 1967, pp. 1-22.

64. Thomann, Robert V., and M. J. Sobel, Estuarine Water Quality Management and Forecasting, ASCE J. Sanitary Eng. Div., Vol. 90, No. SA5, Oct. 1964, Paper 4116, pp. 9-36.

RECEIVED
DEC 18 1969
INPUT SECTION
CLEARINGHOUSE

WATER RESOURCES RESEARCH CENTER
VIRGINIA POLYTECHNIC INSTITUTE BLACKSBURG, VIRGINIA 24061